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Two Submatrix Properties of Certain Induced Norms

Charles R. Johnson*

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

(May 7, 1975)

Induced matrix norms N, defined consistently for all orders up to a given order, which have the property:

$$N(\hat{A}) \leq N(A)$$

for all principal submatrices $\hat{A}$ of an arbitrary $m \times m$ complex matrix A are characterized. Conditions are also given which insure that

$$N(A \oplus B) = \max \{N(A), N(B)\}$$

Key words: Consistency; matrix norm; monotonicity; principal submatrix; spectral radius, vector norm.

It is well known that for $A \in M_k(C)$, the $k \times k$ complex matrices,

$$\rho(A) \leq \| A \| \tag{1}$$

where $\rho(A)$ is the *spectral radius* of A and $\| \cdot \|$ is any vector-norm-induced *matrix norm*. However, if $\hat{A}$ is a principal submatrix of A, it is *not* in general the case that

$$\rho(\hat{A}) \leq \rho(A), \tag{2}$$

nor is it the case that

$$\rho(\hat{A}) \leq \| A \|. \tag{3}$$

EXAMPLE 1: Let $\| \cdot \|_\infty$ be the infinity vector norm on C^2, $\| x \|_\infty \equiv \max \{|x_1|, |x_2|\}$ and define

$$\| x \|_S \equiv \| Sx \|_\infty$$

where

$$S = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}.$$

Then the vector norm $\| \cdot \|_S$ induces a matrix norm $\| \cdot \|_S$ on $M_2(C)$ which is related to the matrix norm

$$\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \|_\infty \equiv \max \{|a| + |b|, |c| + |d|\}$$

by

$$\| A \|_S = \| SAS^{-1} \|_\infty.$$

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* Present address: Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Md. 20742.

Now, if

$$A=\begin{pmatrix} a & a \\ -a & -a \end{pmatrix},$$

where $a>0$ then

$$\rho(A)=0$$

while

$$\rho(\hat{A})=a$$

where $\hat{A}$ is either principal submatrix of A. However,

$$||A||_S=\frac{1}{2}a$$

so that

$$\rho(\hat{A}) \nleq \rho(A) \text{ and } \rho(\hat{A}) \nleq \|A\|_S.$$

Also, the spectral radius enjoys the property that

$$\rho(A\oplus B)=\max\{\rho(A),\rho(B)\} \tag{4}$$

whereas *not* all matrix norms $\|\cdot\|$ satisfy

$$\|A\oplus B\|=\max\{\|A\|,\|B\|\}. \tag{5}$$

For example, the Euclidean matrix norm $\|A\|=(\Sigma|a_{ij}|^2)^{1/2}$ fails condition (5). However, the matrix norms induced by the p-norms

$$\|x\|_p=\left(\sum_{i=1}^{k}|x_i|^p\right)^{1/p} \tag{6}$$

on vectors do satisfy (3) and (5). It is the goal of this note to point out general characteristics of classes of induced matrix norms which insure the validity of (3) or (5).

By a *vector norm* we mean a map $\|\cdot\|: C^k \to R$, the real numbers, which satisfies for all $x, y \in C^k$

$$\|x\|\geq 0, \text{ and } =0 \Longleftrightarrow x=0; \tag{7}$$

$$\|cx\|=|c|\,\|x\| \qquad \text{for any complex number } c; \tag{8}$$

and

$$\|x+y\|\leq\|x\|+\|y\|. \tag{9}$$

By a *matrix norm* we mean a map $\|\cdot\|: M_k(C)\to R$, which satisfies for all $A, B\in M_k(C)$

$$\|A\|\geq 0, \text{ and } =0 \Longleftrightarrow A=0; \tag{10}$$

$$\|cA\|=|c|\,\|A\| \qquad \text{for any complex number } c; \tag{11}$$

$$\|A+B\|\leq\|A\|+\|B\| \tag{12}$$

$$\|AB\|\leq\|A\|\;\|B\|. \tag{13}$$

Any vector norm $\|\cdot\|$ on C^k uniquely induces a matrix norm $\|\cdot\|$ on $M_k(C)$ by

$$\|A\|\equiv\max_{\|x\|=1}\|Ax\| \tag{14}$$

Finally, because of the homogeneity (8) it is clear that a vector norm $\|\cdot\|$ is precisely determined by its *unit ball*

$$B_{\|\cdot\|} \equiv \{x \epsilon C^k : \| x \| \leq 1\} \tag{15}$$

which is well known to be a convex, compact set containing the origin.

Since we compare the norm of a matrix to norms of principal submatrices, we need to consider classes of norms defined for all orders less than or equal to a given order.
Let

$$n \equiv \{n_k, k=1, \ldots, m : n_k \text{ is a vector norm on } C^k\} \tag{16}$$

and if $x \epsilon C^k$, $k \leq m$, we abbreviate $n_k(x)$ by $n(x)$. Also let N_k be the matrix norm which n_k induces on $M_k(C)$ via (14) and

$$N \equiv \{N_k : k=1, \ldots, m\}. \tag{17}$$

If $A \epsilon M_k(C)$, $k \leq m$, we abbreviate $N_k(A)$ by $N(A)$.

DEFINITION 1: *We say that* n *is consistent if for each* $k=2, \ldots, m$ *and each* $j=1, \ldots, k$

$$n_k(x_1, \ldots, x_{j-1}, 0, x_{j+1} \ldots, x_k) = n_{k-1}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k). \tag{18}$$

We further say that n *is strongly consistent if for all* $k=2, \ldots, m$ *and all* $x^i \epsilon C^{k_i}$, $\sum_{i=1}^{l} k_i = k$, *we have*

$$n_k(x^1, \ldots, x^l) \equiv n_l(\| x^1 \|, \ldots, \| x^l \|). \tag{19}$$

REMARK 1: The norm set n is consistent if and only if B_{n_k} is the intersection of any k-dimensional coordinate plane with $B_{n_{k+1}}$, $k=1, \ldots, m-1$.

DEFINITION 2: *We say that* n_k *is monotone if for* $x=(x_1, \ldots, x_k)$ *and all* $j=1, \ldots, k$

$$n_k(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_k) \leq n_k(x_1, \ldots, x_k). \tag{20}$$

We say that n *is monotone if and only if* n_k *is monotone*, $k=2, \ldots, m$.

REMARK 2: From the convexity of B_{n_k} it follows that if n_k is monotone then

$$n_k(x_1, \ldots, x_{j-1}, \alpha x_j, x_{j+1}, \ldots, x_k) \leq n_k(x_1, \ldots, x_k) \tag{21}$$

for $0 \leq \alpha \leq 1$.

REMARK 3: Geometrically, it is clear that n_k is monotone if and only if B_{n_k} satisfies "$x \epsilon$ boundary (B_{n_k}) implies that the projection of x onto any $k-1$ dimensional coordinate plane is in B_{n_k}". Moreover, if n is consistent, then n is monotone if an only if $x \epsilon$ boundary (B_{n_m}) implies that the projection of x onto any k dimensional coordinate plane, $k < m$, is in B_{n_m}.

EXAMPLE 2: Each of the p-norms defined by (6) is consistent, strongly consistent and monotone.

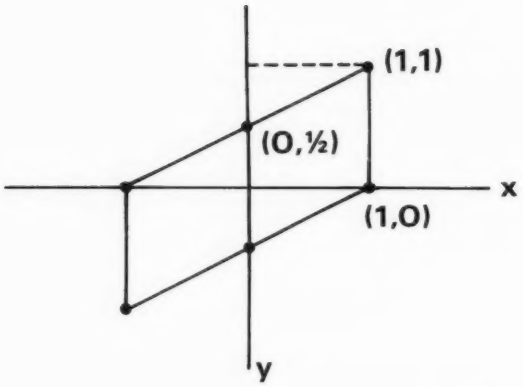
EXAMPLE 3: If $\|\cdot\|_1$ is the p-norm with $p=1$ in (6) on C^2 and we define

$$\| (x, y) \|_T = \| T \begin{pmatrix} x \\ y \end{pmatrix} \|_1$$

where

$$T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

then $B_{\|\cdot\|_T}$ is



In this case $\| (0, 1) \|_T = 2 \not\leq 1 = \| (1, 1) \|_T$, and $\| \cdot \|_T$ is *not* monotone. Visually, the projection of (1, 1) from the boundary of $B_{\|\cdot\|_T}$ onto the y axis leaves $B_{\|\cdot\|_T}$.

Our first main result concerns norms of principal submatrices.

THEOREM 1: *If* n *is consistent and* $\hat{A}$ *denotes any principal submatrix, we have:*

$$N(\hat{A}) \leq N(A) \tag{22}$$

for all $A \in M_k(C), 2 \leq k \leq m$, *if and only if* n *is monotonic.*

PROOF: First of all, suppose n is consistent and monotonic. Because of the consistency it suffices to suppose $\hat{A} = A_1$, that principal submatrix determined by deleting the first row and column of A. (The argument for any other case would be entirely analogous.) We then have

$$N(A_1) = \max_{n(x)=1} n(A_1 x) \qquad \text{by (14)}$$

$$= \max_{n(0,\,x)=1} n\left(\begin{pmatrix} 0 & 0 \\ 0 & A_1 \end{pmatrix}\begin{pmatrix} 0 \\ x \end{pmatrix}\right) \qquad \text{by (18)}$$

$$= \max_{n(y)=1} n\left(\begin{pmatrix} 0 & 0 \\ 0 & A_1 \end{pmatrix} y\right) \qquad \text{by (20)}$$

$$= N\left(\begin{pmatrix} 0 & 0 \\ 0 & A_1 \end{pmatrix}\right) = N\left(\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} A \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\right)$$

$$\leq N\left(\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\right) N(A) N\left(\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\right) \qquad \text{by (13)}$$

$$= 1\, N(A)\, 1 \qquad \text{by (20)}$$

$$= N(A),$$

which completes the first portion of the proof.

Conversely, suppose that n is consistent and that $N(\hat{A}) \leq N(A)$ for all $A \epsilon M_k(C), 2 \leq k \leq m$. This implies that

$$N\left(\begin{pmatrix} I & 0 & 0 \\ 0 & & \\ 0 & & I \end{pmatrix}\right) = 1 \tag{23}$$

since

$$1 \leq N\left(\begin{pmatrix} I & 0 & 0 \\ 0 & & \\ 0 & & I \end{pmatrix}\right) \leq N(I) = 1$$

which by definition means that

$$\frac{n(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots x_k)}{n(x)} \leq 1 \tag{24}$$

or, equivalently, that n is monotonic (20). This completes the proof.

COROLLARY 1: *If* n *is consistent and monotonic, then for any* $A \epsilon M_m(C)$

$$\rho(\hat{A}) \leq N(A) \tag{25}$$

for all principal submatrices $\hat{A}$ *of* A.

PROOF: That $\rho(\hat{A}) \leq N(\hat{A})$ is always the case if N is induced. From theorem 1, $N(\hat{A}) \leq N(A)$.

THEOREM 2: *If* n *is strongly consistent and monotone, and* $A \epsilon M_{m_1}(C)$, $B \epsilon M_{m_2}(C)$, $m_1 + m_2 = m$, *then*

$$N(A \oplus B) = \max \{N(A), N(B)\}. \tag{26}$$

PROOF: Suppose $x \epsilon C^{m_1}$ and $y \epsilon C^{m_2}$ satisfy $n(x) = n(y) = 1$ and $n(Ax) = \alpha$, $n(By) = \beta$. Then $tx \oplus (1-t)y \epsilon C^m$ and $(A \oplus B)(tx + (1-t)y) = tAx \oplus (1-t)By$ so that

$$\frac{n((A \oplus B)(tx \oplus (1-t)y))}{n(tx \oplus (1-t)y)} = \frac{n((t\alpha, (1-t)\beta))}{n((t, (1-t)))} \tag{27}$$

since n is strongly consistent. Since n is monotone, it follows from remark 2 that, for $0 \leq t \leq 1$, the maximum attained by the right-hand side of (27) is the greater of α and β (when either $t = 1$ or $t = 0$). We may thus conclude that

$$\max_{n(z)=1} (A \oplus B)z \equiv N(A \oplus B)$$

is attained when z is of the form either

$$z = z^1 \oplus 0 \text{ or } z = 0 \oplus z^2,$$

$z^1 \epsilon C^{m_1}$, $z^2 \epsilon C^{m_2}$. Then $N(A \oplus B)$ is either $N(A)$ or $N(B)$, whichever is larger, as was to be shown.

By an entirely similar approach, Theorem 2 may be generalized to:

COROLLARY 2: *If* n *is strongly consistent and monotone, and* $A_i \epsilon M_{m_i}(C)$, $i = 1, \ldots, l$ *where* $m_1 + \ldots + m_l = m$, *then*

$$N(A_1 \oplus \ldots \oplus A_l) = \max_{1 \leq i \leq l} N(A_i). \tag{28}$$

As noted previously, the p-norms (6) are consistent, strongly consistent and monotone. It would be of further interest to determine which vector norms of the type (16) are consistent, strongly consistent and/or monotone.

Author's Note (added in proof):

Theorem 2 of our note should be compared with a result of a similar nature (theorem 2) contained in "Norms on Direct Sums and Tensor Products," Math. Comp. **26,** 401–414 (1972). In fact it follows from the present work together with that paper (whose definitions we follow for new terms) that the following conditions on a consistent vector norm n are equivalent:

(i) n is monotone;
(ii) n is absolute;
(iii) the matrix norm N induced by n satisfies (22); and
(iv) the matrix norm N induced by n satisfies (26).

(Paper 79B3&4–424)

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Properties of Neighboring Sequences in Stratifiable Spaces*

Ralph R. Sabella**

(May 12, 1975)

In T_0-spaces metrizability can be characterized in terms of mutual convergence of "neighboring sequences." In this paper Nagata spaces are characterized in terms of a convergence property of neighboring sequences and more generally it is shown that in all stratifiable spaces, neighboring sequences satisfy a similar convergence property.

Key words: Coconvergent; contraconvergent; Nagata spaces; open neighborhood assignments; stratifiable spaces; U-linked sequences

An open neighborhood assignment (ONA) is defined in [4][1] as a function

$$U : X \times Z \to \{N(x) : x \in X\}$$

such that $x \in U(x, n) \equiv U_n(x)$ where X is a topological space, Z is the set of natural numbers and $N(x)$ is the collection of open neighborhoods of x. If U is an ONA then the sequence $\{x_n\}$ is *U-linked* to $\{y_n\}$ if $x_n \in U_n(y_n)$ for all n. Using the notation $Cp\{x_n\}$ for the set of cluster points of $\{x_n\}$ a space will be called *coconvergent* (*contraconvergent*) if $Cp\{x_n\} \subset Cp\{y_n\}$ ($Cp\{y_n\} \subset Cp\{x_n\}$) whenever $\{y_n\}$ is U-linked to $\{x_n\}$. If on X there is an ONA U satisfying some property P I shall say "X is P" or "U is P." Also, U will be said to be nested if $U_{n+1}(x) \subset U_n(x)$ for all x and n.

It was proved in [4]:

THEOREM 1: X *is metrizable iff it is a coconvergent, contraconvergent* T_0*-space.*

Also, examples of a T_2*, coconvergent space and a* T_2*, contraconvergent space, neither of which are metrizable were given in* [4]. *Coconvergence implies first countability whereas nonfirst countable contraconvergent spaces exist.*

R. W. Heath in [3] proved:

THEOREM 2: *A* T_1*-space* X *is a Nagata space* (*first countable and stratifiable*) *iff there is an ONA* U *on* X *such that* (*a*) U *is first countable and* (*b*) *for every* $x \in X$ *and open set* R *containing* $x \in X$ *there is an* $n \in Z$ *such that* $U_n(x) \cap U_n(y)$ *implies* $y \in R$.

(*Note: Condition* (*a*) *is implied by* (*b*): *If* $x \in X$ *and* $R \in N(x)$ *such that for all* n *there is a* $y_n \in U_n(x) - R$, *then* $U_n(x) \cap U_n(y_n) \neq 0$. *It follows from* (*b*) *there is a* $y_k \in R$ *for some* k *contradicting the way the* y_n *were chosen.*)

PROPOSITION 3: *A* T_1*-space* X *is a Nagata space iff it is first countable and contraconvergent.*

PROOF: Let U be an ONA on X satisfying the conditions of Theorem 2. Without loss of generality we may take U to be nested. Let $\{y_n\}$ be U-linked to $\{x_n\}$ with $y \in Cp\{y_n\}$. Given an $R \in N(y)$ and $N \in Z$ there is an $n_1 > N$ such that $U_{n_1}(y) \subset R$. Also, since U is nested there is an $n_2 > n_1$ such that if x satisfies $U_{n_2}(x) \cap U_{n_2}(y) \neq 0$ then $x \in R$. Finally, there is a $k > n_2$ such

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**Present address: California State University, Northridge, California 91324.
[1] Figures in brackets indicate the literature references at the end of this paper.

that $y_k \in U_{n_2}(y) \subset U_{n_1}(y)$. Since $\{y_n\}$ is U-linked to $\{x_n\}$, $y_k \in U_k(x_k) \cap U_{n_2}(y)5 \subset U_{n_2}(x_k) \cap U_{n_2}(y)$. It now follows from $k > N$ $y \in Cp\{x_n\}$ and that U is contraconvergent.

Conversely, without loss of generality it may be assumed there is an ONA U on X which is contraconvergent and first countable. If U does not satisfy the condition of Theorem 2, then for some x and $R \in N(x)$ there are sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \in U_n(x) \cap U_n(y_n)$ and $y_n \notin R$ for all n. It follows from the first countability and contraconvergence of U that $x \in Cp\{y_n\}$ which is a contradiction.

COROLLARY 4: *A T_0-space is metrizable iff it is coconvergent and stratifiable.*

Coconvergent spaces are those included in the class of spaces in which compact sets have countable local bases (D_0-spaces) [5]. The set of all ordinals less than or equal to the first uncountable ordinal, with the order topology is an example of a D_0-space which is not coconvergent. It was shown in [5] that spaces in which the stratifications of open sets satisfy a certain monotone condition, "coconvergent" can be replaced by "D_0" in Corollary 4.

If X is stratifiable, I will use: R_n, $(X - F)_n$ and $[U_k(x)]_n$ to denote the nth layers of stratifications of the open sets R, $X - F$ and $U_k(x)$ respectively and more generally, A^- for the closure of A. Without loss of generality I shall assume $R_n \subset R_{n+1}$ for any open R. It will be shown that if X is stratifiable that the ONA U defined by $U_n(x) = X - (X - x_n)^-$ satisfies a condition similar to that defining contraconvergent spaces. The ONA U will be referred to as the ONA associated with the given stratification.

DEFINITION 5: *An ONA U on X satisfies property A if whenever $\{y_n\}$ is U-linked to $\{x_n\}$ and $y \in Cp\{y_n\}$ then for any $N \in Z$ there is a $k > N$ such that $x_k \in U_N(y)$.*

PROPOSITION 6: *A T_1-space is a Nagata space iff there is a first countable ONA, U on X satisfying property A.*

PROOF: If X is a Nagata space, by Proposition 3 there is an ONA, U which is first countable and contraconvergent. If $\{y_n\}$ is U-linked to $\{x_n\}$ and $y \in Cp\{y_n\}$ then $y \in Cp\{x_n\}$ and it follows that U satisfies condition A.

Let U be a first countable ONA satisfying condition A. If $\{y_n\}$ is U-linked to $\{x_n\}$ with $y \in Cp\{y_n\}$ and if $N_1 \in Z$ and $R \in N(y)$ then there is an $N_2 > N_1$ and by property A a $k > N_2$ such that $x_k \in U_{N_2}(y) \subset R$ proving $y \in Cp\{x_n\}$ and that U is contraconvergent. By Proposition 3, X is a Nagata space.

PROPOSITION 7: *Let X be a stratifiable space. Then the ONA U associated with a given stratification satisfies property A.*

PROOF: It follows from $R_n \subset R_{n+1}$ for any open R that U is nested. If U does not satisfy property A, then there exist a $\{y_n\}$ U-linked to $\{x_n\}$, a $y \in Cp\{y_n\}$ and an $N \in Z$ such that for all $k > N$, $x_k \notin U_N(y)$. Hence $F \equiv \{x_k : k > N\}^- \subset X - U_N(y)$. For all $n \in Z$, $[U_N(y)]_n^- \subset (X - F)_n^-$. Also, for all $k > N$, $x_k \in F$ and for all $n \in Z$, $(X - F)_n^- \subset (X - x_k)_n^-$. Hence for all $k > N$ and $n \in Z$, $U_n(x_k) \subset X - (X - F)_n^-$. Since $y \in U_N(y)$, it follows $y \in [U_N(y)]_M$ for some M. Therefore there is an $r \in Z$ with $r > \max(N, M)$ such that $y_r \in [U_N(y)]_M$. On the other hand we have $y_r \in U_r(x_r) \subset U_M(x_r) \subset X - (X - F)_M^- \subset X - [U_N(y)]_M^-$ which is a contradiction.

In the next proposition I use Ceder's result [2] that locally compact M_3 ($\equiv$ stratifiable) spaces are metrizable.

PROPOSITION 8: *A locally compact space is metrizable iff it is contraconvergent and Hausdorff.*

PROOF: Let U be a contraconvergent ONA on X and for each x let $C(x)$ be a neighborhood of x whose closure is compact. Also for each x and n let $U_n'(x) = U_n(x) \cap C(x)$. Suppose $R \in N(x)$ and for all n there is a $y_n \in U_n'(x) - R$. Then there is a $y \in [C(x)]^- \cap Cp\{y_n\}$. It follows that $y = x$ which is a contradiction. Hence U' is first countable and by Proposition 3 and Ceder's result, X is metrizable.

The converse is immediate from Theorem 1.

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Maximin Facility Location*

Peter J. Slater

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

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Using the criterion that vertex u of a graph G is "more central" than vertex v if there are more vertices closer to u than to v, the security center and security centroid of a graph are defined. Several results, including the fact that the security center of a tree is the centroid, are presented. A simple algorithm to find the security center (centroid) of a tree is presented. The problem of finding the security center of G is shown reducible to a search over a single block of G.

Key words: Center; centroid; facility location; graph; tree.

1. Introduction

Consider a finite, connected, undirected graph G with vertex set V and edge set E. If $e \epsilon E$, then $L(e)$ will denote the length of edge e $(L(e) > 0)$, and if P is a path, then $L(P) = \Sigma_{e \epsilon P} L(e)$ denotes the length of P. The distance between vertices u and v, denoted $d(u, v)$, is the length of the shortest path from u to v.

Several mathematical problems arise from different instances of the question of what is an optimal location for a facility in a graph. Hakimi [3][1] posed and considered two such optimization problems. The first problem is to determine a vertex $u \epsilon V$ so as to minimize

$$D_1(u) = \Sigma_{v \epsilon V} w(v) d(v, u) \tag{1}$$

where $w(v)$ is a nonnegative weight given for each $v \epsilon V$. The second problem is to determine a vertex $u \epsilon V$ so as to minimize

$$D_2(u) = \max\{w(v)d(u, v) : v \epsilon V\}. \tag{2}$$

Interpretations of these problems [2] include minimizing transportation cost to a central facility for (1), and minimizing a maximum response time from an emergency facility, such as a hospital, for (2). While a more general treatment is presented in the third section, it is assumed for now that $w(v) = 1$ for every $v \epsilon V$. Until section four, it is assumed that each edge has length equal to one.

For each $u \epsilon V$, let S_u denote the family of distances from u to the other vertices in V. That is, $S_u = \{d(u, v) : v \epsilon V - \{u\}\}$. Note that one allows repetitions in S_u. For example, for the graph G_1 of figure 1, $S_{u_1} = \{1, 1, 2, 2, 3\}$ and $S_{u_2} = \{1, 1, 1, 1, 2\}$.

Hakimi's first problem involves finding a vertex u for which the sum of all the elements of S_u is a minimum. This sum will be called the *distance of u in G*, denoted $d(u)$. The set of all vertices

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[1] Figures in brackets indicate the literature references at the end of this paper.

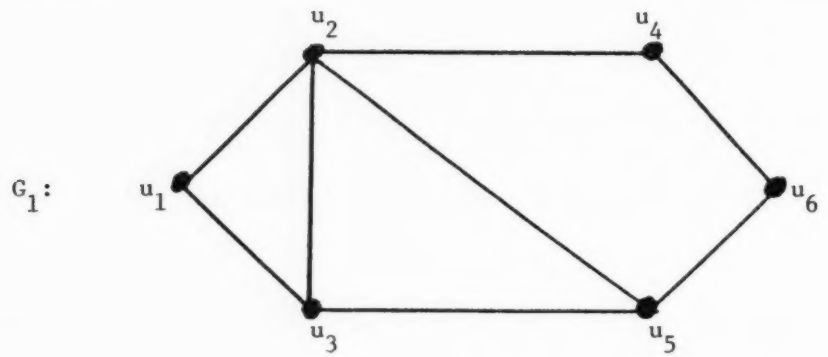


FIGURE 1. *Graph* G_1.

with minimum distance in G is called the *centroid*, and Zelinka [6] has shown that for trees this agrees with the "branch weight" definition of centroids (see sec. 2). The maximum value in S_u will be called the eccentricity of u, denoted $e(u)$, and Hakimi's second problem is to find a point of minimum eccentricity. The set of all vertices with minimum eccentricity is called the *center*.

Another set of numbers will now be associated with u, and paralleling the previous definitions, the security centroid and security center will be defined for G. Consider two players X and Y (e.g., two companies or two countries), each of which is to locate a facility at a vertex of G (e.g., a store or a military base), say at vertices x and $y (x \neq y)$. If $d(v, x) < d(v, y)$ then x will be said to have more "influence" over v than y does (customers usually shop at the nearer of two otherwise equal stores, a location is more easily protected by the closer of two bases). The player who must choose a location first desires to have more influence than his competitor over as many vertices as possible.

If u and v are distinct vertices in V, let V_{uv} be the set of vertices which are closer to u than to v. That is,

$$V_{uv} = \{s \epsilon V : d(s, u) < d(s, v)\}.$$

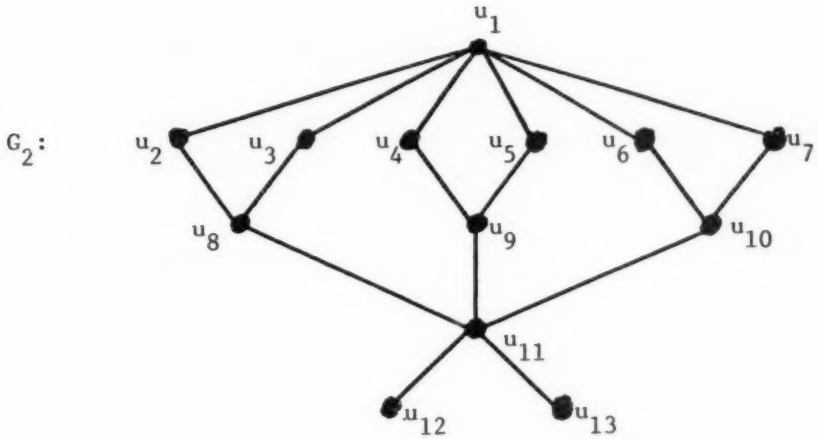
Note that $u \epsilon V_{uv}$. Let $|S|$ denote the cardinality of a set S, and let $f(u, v) = |V_{uv}| - |V_{vu}|$. That is, $f(u, v)$ equals the number of vertices closer to u than to v minus the number of vertices closer to v than to u. One is now interested in maximizing the functions

$$f(u) = \min\{f(u, v) : v \epsilon V - u\}, \tag{3}$$

$$g(u) = \Sigma_{v \epsilon V - u} f(u, v). \tag{4}$$

Label the vertices of G as $u_1, u_2, \ldots, u_p$. One can form an array M with $m_{ij} = f(u_i, u_j)$ $(1 \leq i, j \leq p, i \neq j)$. Now $f(u_i)$ is the smallest element in the ith row, and $g(u_i)$ is the sum of the ith row. Examples of graphs and some of the rows from the corresponding arrays are given in figures 2, 3, and 4.

If there is a point u such that $f(u) \geq 1$, then, since $f(v, u) = -f(u, v) \leq -1$, $f(v) \leq -1$ for every $v \epsilon V - u$ (as in fig. 2), and so u is the unique maximizer of f. There may be several points $u_1, \ldots, u_k$ such that $f(u_i) = 0$ for $1 \leq i \leq k$ (as in fig. 3); then these points are the maximizers of f.



u \ v	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_{13}
u_1	—	3	3	3	3	3	3	1	1	1	1	4	4
u_2	−3	—	0	0	0	0	0	−1	−1	−1	−3	3	3
u_8	−1	1	1	1	1	1	1	—	0	0	−5	3	3
u_{11}	−1	3	3	3	3	3	3	5	5	5	—	11	11
u_{12}	−4	−3	−3	−3	−3	−3	−3	−3	−3	−3	−11	—	0

FIGURE 2. *Graph* G_2 *with* $f(u_i) > 0$.

Finally (as in fig. 4) it is possible that $f(u) \leq -1$ for every $u \epsilon V$. For this last case, "follow the leader is optimal," that is, the player who chooses a location after the other player has chosen has the advantage. In the first case, the first moving player has the advantage, while in the second case, the order of moving is immaterial.

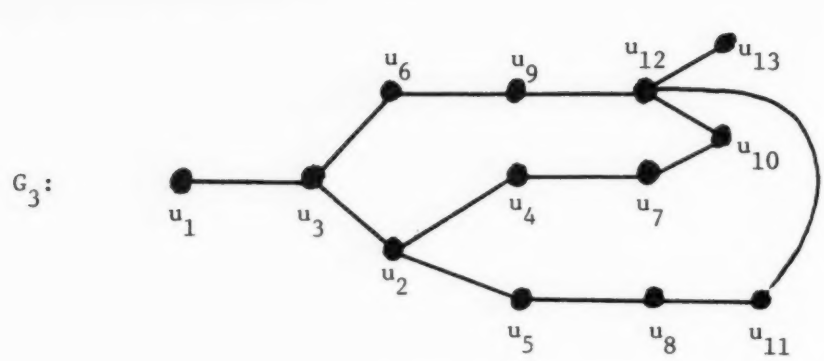
Let $r(G)$, the radius of G, and $s(G)$ be defined as follows.

$$r(G) = \min_{u \epsilon V} (\max_{v \epsilon V} d(u, v)) \tag{5}$$

$$s(G) = \max_{u \epsilon V} (\min_{v \epsilon V - u} f(u, v)) \tag{6}$$

$$= \max_{\mu \epsilon V} f(\mu)$$

The center of G consists of all vertices u for which $e(u)$ equals $r(G)$. The *security center* of G is the set of all vertices u for which $f(u)$ equals $s(G)$, and it will be denoted by $C(G)$. The *security centroid* of G, denoted $C_1(G)$, is the set of all vertices at which function g is a maximum. Referring



u \ v	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_{13}
u_1	—	−6	−11	−5	−5	−5	−3	−3	−3	−3	−3	−4	−1
u_2	6	—	1	3	3	1	2	2	1	1	1	0	3
u_{12}	4	0	1	1	1	2	3	3	3	5	5	—	11
u_{13}	1	−3	−2	−2	−2	−1	−1	−1	−4	−3	−3	−11	—

FIGURE 3. *Graph* G_3 *with* $f(u_2)=f(u_{12})=0$.

to figures 1, 2, 3, and 4, $C(G_1)=C_1(G_1)=\{u_2\}$, $C(G_2)=\{u_1\}$, $C_1(G_2)=\{u_{11}\}$, $C(G_3)=\{u_2, u_{12}\}$, $C_1(G_3)=\{u_{12}\}$, $C(G_4)=\{u_1, u_4, u_5, u_6, u_7, u_{11}, u_{12}, u_{13}\}$, and $C_1(G_4)=\{u_7\}$.

2. Security Centers and Security Centroids of Trees

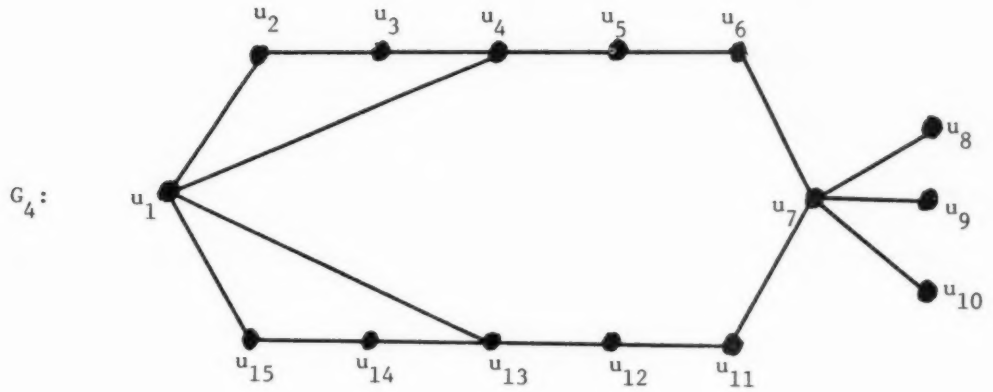
It is reasonable to think that if u is closer to more vertices of G than v is, then u is "more central" than v. However, as for G_4 (fig. 4), it is possible to find a set of vertices $\{u_1, u_2, \ldots, u_k\}$, for example vertices u_4, u_5, u_{13}, and u_{12} in G_4, such that u_i is "more central" than u_{i+1} $(1\leq i\leq k-1)$ and u_k is "more central" than u_1. Despite this, the next theorem is a strong indication that the security center truly is a set of "central" vertices.

Suppose G is a tree and $v\epsilon V$. Label the components of $G-v$ as $B_1, B_2, \ldots, B_t$ (where the degree of v is t). The *weight at point* v is the maximum cardinality of a V_i where V_i is the set of vertices in $B_i(1\leq i\leq t)$. A vertex v is called a *branch weight centroid vertex* of the tree G if v has minimum weight, and the *branch weight centroid* of G consists of all such vertices. As noted previously, Zelinka has shown the branch weight centroid of a tree to be identical with its centroid.

THEOREM 1: *If* G *is a tree, then the security center of* G *is the centroid.*

PROOF: Jordan [5] has shown that the centroid of a tree consists of one vertex or two adjacent vertices.

First, suppose the centroid consists of one point, say v, with branches $B_1, B_2, \ldots, B_t$ whose vertex sets satisfy $|V_1|\leq|V_2|\leq\ldots\leq|V_t|$. Let u_t be the vertex of V_t adjacent to v. If $|V_t|>|V_1|$



u \ v	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}
u_1	–	11	5	−1	0	1	1	3	3	3	1	0	−1	5	11
u_4	1	6	11	–	1	1	1	3	3	3	0	−1	0	5	6
u_5	0	1	5	−1	–	1	0	3	3	3	−1	0	1	4	3
u_{12}	0	3	4	1	0	−1	0	3	3	3	1	–	−1	5	1
u_{13}	1	6	5	0	−1	0	1	3	3	3	1	1	–	11	6

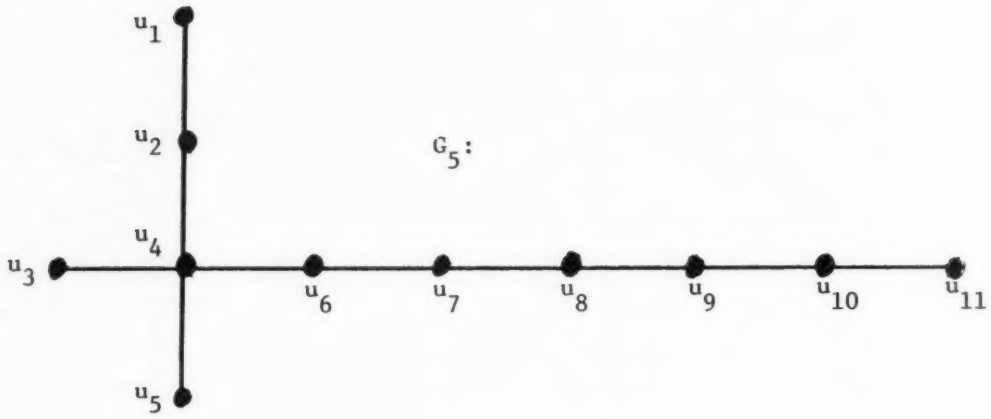
FIGURE 4. *Graph* G_4 *with* $f(u) \leq -1$ *for every vertex* u.

$+|V_2|+ \ldots +|V_{t-1}|$, then u_t has a weight bounded above by max $(|V_1|+ \ldots +|V_{t-1}|, |V_t| - 1) \leq |V_t|$. Since the weight of u_t is less than or equal to the weight $|V_t|$ of v, the centroid of G could not be $\{v\}$. Thus $|V_t| \leq |V_1|+ \ldots +|V_{t-1}|$. Now let $u \in V_i$. Then $f(v, u) \geq (\Sigma_{j=1, j\neq i}^{t} |V_j|) -(|V_i|-1) \geq 1$. Thus $f(v) \geq 1$ and $C(G)=\{v\}$ is the centroid.

Next, suppose the centroid of G consists of two adjacent vertices, v_1 and v_2. Let B_1 and B_2 be the components of $G-v_1v_2$, with $v_i \in B_i$ and V_i the vertex set of B_i $(i=1, 2)$. If $|V_1| > |V_2|$ then the weight of v_2 is $|V_1|$, and the weight of v_1 is bounded above by max $(|V_2|, |V_1|-1) < |V_1|$. Thus V_1 and V_2 would have different weights, so the centroid of G could not be $\{v_1, v_2\}$. Similarly $|V_2| \leq |V_1|$. Thus $|V_1| = |V_2|$. If $u \in V_i - \{v_i\}$ $(i=1$ or $2)$, then $f(u, v_i) \leq -2$, which implies $f(u) \leq -2$. It is easy to see that $f(v_1)=f(v_2)=0$, and so $C(G)=\{v_1, v_2\}$ is the centroid.

COROLLARY 1a: *The security center of a tree consists of one point or two adjacent points.*

Graph G_4 (fig. 4) is an example of a graph for which the center, centroid, security center and security centroid are different subsets of V. Even for trees it is possible to have the center, centroid (security center) and security centroid distinct, as in figure 5. Graph G_6 (fig. 6) has a security centroid consisting of three independent points, namely those which are adjacent to the endpoints, when $k \geq 14$.



u \ v	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}
u_4	8	7	9	—	9	−1	0	1	2	3	4
u_6	7	4	5	1	5	—	1	2	3	4	5
u_7	4	1	1	0	1	−1	—	3	4	5	6

FIGURE 5. *Graph* G_5 *with center* $\{u_7\}$, *centroid* $\{u_6\}$ *and security centroid* $\{u_4\}$.

One can modify graph G_6 by using n branches at v (instead of three), each of which is identical to a branch at v in G_6. It can be shown that if k is sufficiently large, then the security centroid is precisely those n points which are adjacent to endpoints. One therefore has the following theorem.

THEOREM 2: *For any natural number* n *there is a tree* T_n *whose security centroid consists of* n *independent vertices* (*in fact, no three of which are on one path*).

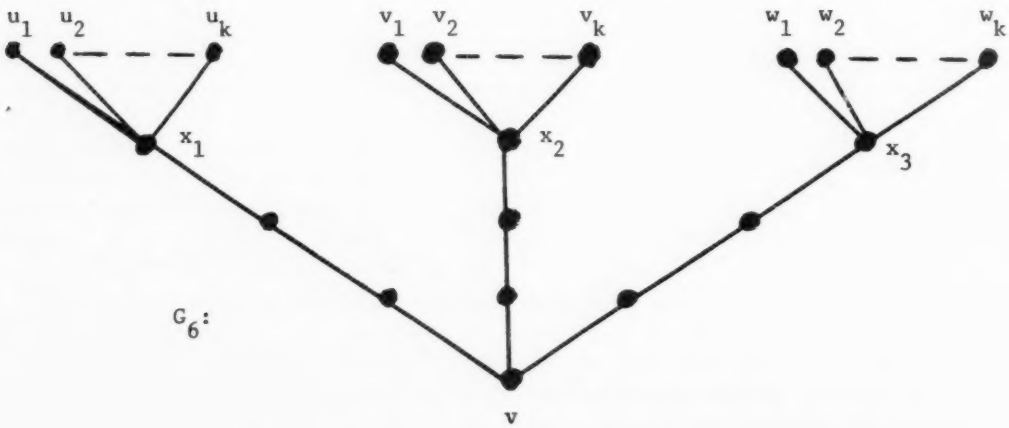


FIGURE 6. *Graph* G_6 *with security centroid* $\{x_1, x_2, x_3\}$.

3. Cutpoints and Bridges in Graphs

As Goldman has done for (2) in [2], it is convenient to put function f from (3) in a more general setting by assigning a nonnegative integer valued addend $a(s)$ to each vertex s. One may think of $a(s)$ in this case as a number of vertices of degree one, each of which is adjacent to s. If $s\epsilon V_{uv}$ then each of these vertices would also be closer to u than to v. Let $\alpha(s)=a(s)+1$, and let $f^*(u, v)$ be defined as follows.

$$f^*(u, v) = (\,|\,V_{uv}\,|+\Sigma_{s\epsilon V_{vu}}a(s)) - (\,|\,V_{vu}\,|+\Sigma_{s\epsilon V_{vu}}a(s))$$

$$= \Sigma_{s\epsilon V_{uv}}\alpha(s) - \Sigma_{s\epsilon V_{uv}}\ \alpha(s). \tag{7}$$

Note that $u\epsilon V_{uv}$. One is now interested in maximizing the function

$$f^*(u) = \min\{f^*(u, v) : v\epsilon V\}. \tag{8}$$

Maximizing f is the same as maximizing f^* when $\alpha(s)=1$ for every vertex s of V. The security center of (G, α) is the set of vertices for which f^* is a maximum, and is denoted $C^*(G, \alpha)$.

Given a connected graph G, a *cutpoint* (*bridge*) is a vertex (edge) whose removal disconnects G. A block B of G is a connected subgraph maximal with respect to the property that it has no cutpoints (and therefore if B has at least three vertices, B will also not have a bridge). In this section it will be shown how maximizing the function (8) can be reduced to an equivalent analogous problem for a single block of G. This will result in a simple algorithm for finding the security center (centroid) of a tree.

Suppose one is given (G, α) where $\alpha(u) \geq 1$ for every vertex u of G. Suppose s is a cutpoint, and let $S_1, S_2, \ldots, S_k (k \geq 2)$ denote the vertex sets of the components of $G-s$. Let $\sigma_i=\Sigma_{u\epsilon S_i}\alpha(u)$, and assume the components are labeled in such a way that $\sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_k$.

LEMMA 3: *If* $\sigma_k < \alpha(s)+\Sigma_{i=1}^{k-1}\sigma_i$, then $C^*(G, \alpha)=\{s\}$.

PROOF: Assume $1 \leq j \leq k-1$ and $v\epsilon S_j$. Now $f^*(s, v) \geq (\alpha(s)+\Sigma_{i=1, i\neq j}^{k}\sigma_j)-\sigma_j \geq 1$. Therefore $f^*(s) \geq 1$, implying $f^*(u) \leq -1$ for every vertex u of $G-s$. Thus $C^*(G, \alpha)=\{s\}$.

LEMMA 4: *If* $\sigma_k \geq \alpha(s)+\Sigma_{i=1}^{k-1}\sigma_i$, *then* $C^*(G, \alpha)$ *is contained in* $S_k \cup \{s\}$.

PROOF: If $f^*(s) \geq 1$ then the security center is $\{s\}$. Suppose $f^*(s) \leq 0$. Let $S=S_k \cup \{s\}$, and let $\bar{S}=\cup_{i=1}^{k-1}S_i$. Since $\sigma_k \geq \alpha(s)+\Sigma_{i=1}^{k-1}\sigma_i$, if $w\epsilon\bar{S}$ then $f^*(s, w) \geq 1$. Thus there is a point $x\epsilon S_k$ such that $f^*(s, x)=f^*(s) \leq 0$.

Let $w\epsilon\bar{S}$. Since $f^*(w) \leq f^*(w, x)$, to show that $w\notin C^*(G, \alpha)$ it will suffice to show that $f^*(w, x) < f^*(s, x)=f^*(s)$.

If $v\epsilon V_{xs}$ then $v\epsilon S_k$, and so $d(v, s) < d(v, w)$, implying $v\epsilon V_{xw}$. Thus $V_{xs}\subseteq V_{xw}$. Suppose $v\epsilon V_{wx}$. If $v\notin S_k$ then clearly $v\epsilon V_{sx}$, and if $v\epsilon S_k$ then $d(v, s) < d(v, w) < d(v, x)$, which implies $v\epsilon V_{sx}$. Thus $V_{wx}\subseteq V_{sx}$. Note also that $w\epsilon V_{sx}$. It is claimed that at least one of the two inclusions $V_{xs}\subseteq V_{xw}$ and $V_{wx}\subseteq V_{sx}$ must be proper. For, suppose $V_{wx}=V_{sx}$. In particular, one has $d(s, w) < d(s, x)$ since $s\epsilon V_{sx}$. Hence $d(s, x) \geq 2$. Let $s, s_1, s_2, \ldots, s_p, x$ be a shortest path from s to x, and note that $p \geq 1$. Now p cannot be even, else $s_{p/2}$ would be in V_{sx} but not in V_{wx}. Say $p=2j+1$. Now s_{j+1} is in V_{xw} but not V_{xs}, and the claim is proved.

Now we have

$$\begin{aligned} f^*(w, x) &= \Sigma_{u\epsilon V_{wx}}\alpha(u) - \Sigma_{u\epsilon V_{xw}}\alpha(u) \\ &< \Sigma_{u\epsilon V_{sx}}\alpha(u) - \Sigma_{u\epsilon V_{xs}}\alpha(u) \\ &= f^*(s, x)=f^*(s). \end{aligned}$$

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Consequently $f^*(w) < f^*(s)$, and w is not in the security center.

THEOREM 5: *If* $\sigma_k \geq \alpha(s) + \Sigma_{i=1}^{k-1} \sigma_i$, *then* $C^*(G,\alpha) = C^*(G-\bar{S}, \alpha')$ *where* $\bar{S} = \cup_{i=1}^{k-1} S_i$, $\alpha'(u) = \alpha(u)$ *if* $u \in S_k$, *and* $\alpha'(s) = \alpha(s) + \Sigma_{u \in \bar{S}} \alpha(u)$.

PROOF: Let $S = S_k \cup \{s\}$. By definition of $\alpha'(s)$, if x and y are any two vertices of S then $f^*(x, y)$ in $(G - \bar{S}, \alpha')$ equals $f^*(x, y)$ in (G,α). In particular, if $f^*(s) \geq 1$ in (G, α) then $f^*(s) \geq 1$ in $(G-\bar{S}, \alpha')$, and so $C^*(G,\alpha) = C^*(G-\bar{S}, \alpha') = \{s\}$.

Suppose $f^*(s) \leq 0$ in (G,α). Now $f^*(s) = f^*(s,y)$ in (G,α) for some y in S_k. Thus $f^*(s)$ has the same value in $(G-\bar{S},\alpha')$ as in (G,α). If $x \in S_k$ and $w \in \bar{S}$, then (by the proof of Lemma 4) $f^*(w,x) < f^*(s,x)$, so that $f^*(x,w) = -f^*(w,x) > -f^*(s,x) = f^*(x,s)$. Therefore $f^*(x)$ in (G,α) equals min $\{f^*(x, u): u \in S\}$, which is $f^*(x)$ in $(G-\bar{S}, \alpha')$.

Thus no point of $\bar{S}$ is in $C^*(G,\alpha)$ and every point of $G-\bar{S} = S$ attains the same value of f^* in $(G-\bar{S}, \alpha')$ as in (G,α). Consequently $C^*(G,\alpha) = C^*(G-\bar{S},\alpha')$.

Now suppose edge st is a bridge of G, and let S and T be the vertex sets of the components of $G-st$ with $s \in S$ and $t \in T$. Let $\sigma = \Sigma_{u \in S} \alpha(u)$, and let $\tau = \Sigma_{u \in T} \alpha(u)$. Note that each of s and t is a cutpoint or an endpoint.

COROLLARY 5a: *If* $\sigma = \tau$ *then the security center of* (G, α) *is* $\{s, t\}$.

COROLLARY 5b: *If* $\sigma > \tau$, *then no vertex of* T *can be in the security center.*

COROLLARY 5c: *If* $\sigma > \tau$, *then the security center of* (G, α) *is the security center of* $(G-T, \alpha')$ *with* $\alpha'(u) = \alpha(u)$ *for* $u \in S - \{s\}$ and $\alpha'(s) = \alpha(s) + \Sigma_{v \in T} \alpha(v)$.

The following algorithm, Algorithm *SC*, can be used to reduce the problem of finding $C^*(G, \alpha)$ to the problem of finding $C^*(B, \alpha')$ where B is a single block of G. Given any graph G one finds $C^*(G)$ by setting $\alpha(v) = 1$ for every vertex v.

STEP 1: If G has only one vertex, stop; that vertex is $C^*(G, \alpha)$.

STEP 2: If G has no bridges, then go to Step 3. Otherwise, select any bridge st. Let S and T denote the components of $G-st$ with $s \in S$ and $t \in T$. Compute $\sigma = \Sigma_{u \in S} \alpha(u)$ and $\tau = \Sigma_{u \in T} \alpha(u)$. If $\sigma = \tau$, stop; $C^*(G, \alpha) = \{s, t\}$. If $\sigma \neq \tau$, then one can assume $\sigma > \tau$. Modify G by deleting T and replacing $\alpha(s)$ by $\alpha(s) + \tau$. Go to Step 1.

STEP 3. If G has no cutpoints, stop; G is a block. Otherwise, select any cutpoint s. Let S_1, S_2, . . ., S_k be the vertex sets of the components of $G-s$, let $\sigma_i = \Sigma_{u \in S_i} \alpha(u)$, and assume $\sigma_i \leq \sigma_k (1 \leq i \leq k-1)$. If $\sigma_k < \alpha(s) + \Sigma_{i=1}^{k-1} \sigma_i$, stop; $C^*(G, \alpha) = \{s\}$. Otherwise modify G by deleting each S_i $(1 \leq i \leq k-1)$ and replacing $\alpha(s)$ by $\alpha(s) + \Sigma_{i=1}^{k-1} \sigma_i$. Repeat Step 3.

Note that if G is modified by Step 3, then $S_k \cup \{s\}$ has at least three points and no bridges are created when one deletes each S_i.

If G is a tree then one never enters Step 3. Steps 1 and 2 therefore comprise an algorithm for finding the security center of a tree. This algorithm is similar to one given in [1] by Goldman.

4. Graphs With Edges of Arbitrary Lengths

If s is a cutpoint of graph G, then it is easy to see that Lemma 3 holds independent of the lengths of the edges. If $\sigma_k \geq \alpha(s) + \Sigma_{i=1}^{k-1} \sigma_i$, it is also clear that $C^*(G, \alpha)$ is not changed if one alters the length of an edge in $\bar{S}$ (as defined in Theorem 5).

THEOREM 6: *If* e *is an edge of graph* G *which is not in the block that contains* $C^*(G, \alpha)$, *then* $C^*(G, \alpha)$ *will not be changed if the length of* e *is altered.*

COROLLARY 7a: *If* T *is a tree, then the security center (the centroid) is determined independent of the lengths of the edges.*

COROLLARY 7b: *Algorithm* SC *can be used to reduce the problem of finding* $C^*(G, \alpha)$ *to the problem of finding* $C^*(B, \alpha')$, *where* B *is a single block of* G, *without considering the lengths of the edges of* G.

Given (G, α), where G is a block, it is necessary to consider the lengths of the edges of G in order to determine $C^*(G, \alpha)$. For example, consider graph G_7 (fig 7) with $\alpha(u_1) = \alpha(u_2) = \alpha(u_3) = 1$

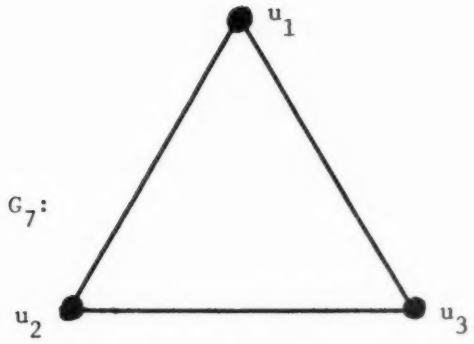

FIGURE 7. *Graph* G_7.

and $L(u_3u_1)=L(u_3u_2)=1$. If $L(u_2u_1)=1$, then $C^*(G,\alpha)=\{u_1,u_2,u_3\}$; if $L(u_2u_1)=2$, then $C^*(G,\alpha)=\{u_3\}$.

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Similarity of Partitioned Matrices*

Robert B. Feinberg**

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

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Suppose that A, B, and T are matrices of order $r\times r$, $s\times s$, and $r\times s$ respectively over a field F. We prove that $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix}$ is similar to $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ iff $AX-XB=T$, for some matrix X. We also give some corollaries and a simple generalization.

Key words: Matric equation; partitioned matrix; rational canonical form; similarity.

Suppose that A, B, and T are matrices of order $r\times r$, $s\times s$, and $r\times s$ respectively over a commutative ring Φ. Let I_n denote the identity matrix of order n. If there is a matrix X of order $r\times s$ over Φ such that $AX-XB=T$, then it is a simple computation that

$$\begin{bmatrix} I_r & X \\ 0 & I_s \end{bmatrix}^{-1} \quad \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \quad \begin{bmatrix} I_r & X \\ 0 & I_s \end{bmatrix} = \begin{bmatrix} I_r & -X \\ 0 & I_s \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \quad \begin{bmatrix} I_r & X \\ 0 & I_s \end{bmatrix} = \begin{bmatrix} A & T \\ 0 & B \end{bmatrix}.$$

Thus $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix}$ is similar to $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ over Φ.

The main result in this paper (Theorem 6) is the converse to the above statement in the case when Φ is a field F, namely, if $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix}$ is similar to $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ over F, then there is a matrix X such that $AX-XB=T$. We also give some corollaries and a simple generalization of the theorem.

This result has been proven independently in [2],[1] and special cases of it have been established in [3], [4], and [6].

At this point we record some notation used throughout the paper. For integers r and s, let F_{rs} denote the collection of $r\times s$ matrices over F and let F'_{rr} denote the group of nonsingular matrices of order r. For $M, N\epsilon F_{rr}$, $M\tilde{S}N$ ($M\tilde{E}N$) represents the statement that M is similar (equivalent) to N over F. We denote the minimal polynomial of M by $f_M(x)$, and the companion matrix of $f_M(x)$ by $C(f_M(x))$. The rational canonical form of M is represented by $RF(M)$, and the minor obtained by deleting row i and column j is represented by $(M)_{ij}$. When the matrix M under discussion is understood, we let R_i denote the ith row of M and C_j denote the jth column. The elementary row operation of adding α times row j to row i is represented by $R_i\rightarrow R_i+\alpha R_j$.

See [5] for a good reference on matrices.

Let us note from the onset that in proving the main result we may assume w.l.o.g. (without loss of generality) that $A=RF(A)$ and $B=RF(B)$. Supposing that $U\epsilon F'_{rr}$ and $V'\epsilon F'_{ss}$ are such that $UAU^{-1}=RF(A)$ and $VAV^{-1}=RF(B)$, then

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**Present address: Department of Mathematics, Clarkson College of Technology, Potsdam, New York 13676.

[1] Figures in brackets indicate the literature references at the end of this paper.

$$\begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \tilde{S} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \Leftrightarrow \begin{bmatrix} RF(A) & UTV^{-1} \\ 0 & RF(B) \end{bmatrix} \tilde{S} \begin{bmatrix} RF(A) & 0 \\ 0 & RF(B) \end{bmatrix}$$

and

$$AX-XB=T \Leftrightarrow RF(A)(UXV^{-1})-(UXV^{-1})RF(B)=UTV^{-1}.$$

Let us note also that we may assume w.l.o.g. that both A and B are nonzero. If both A and B are zero, the result is trivially true. If one of A and B is zero and the other is a multiple of the identity, the result is again trivially true. If A is zero and B is not a multiple of the identity, set $\tilde{A}=A\dot{+}I=I$, $\tilde{B}=B\dot{+}I$. Then both $\tilde{A}$ and $\tilde{B}$ are nonzero,

$$\begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \tilde{S} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \Leftrightarrow \begin{bmatrix} \tilde{A} & T \\ 0 & \tilde{B} \end{bmatrix} \tilde{S} \begin{bmatrix} A & 0 \\ 0 & \tilde{B} \end{bmatrix},$$

and $AX-XB=T \Leftrightarrow \tilde{A}X-X\tilde{B}=T$. We obtain a similar result when the assumptions on A and B are interchanged. Hence in all cases we may assume w.l.o.g. that both A and B are nonzero.

It is now convenient to present the following well-known result. For an outline of the proof see [5, Ch. III, ex. 6 and 7].

LEMMA 0: *Suppose* $A \in F_{rr}$, $B \in F_{ss}$, *and that* A *and* B *have no eigenvalues in common. Then for all* $T \in F_{rs}$ *there is an* $X \in F_{rs}$ *such that* $T = AX - XB$.

Later on we will note a converse to this lemma.

We begin towards the proof of Theorem 6 by recording three technical lemmas. They contain essentially all the hard work.

LEMMA 1: *Suppose* $A_i \in F_{u_i u_i}$, $1 \leq i \leq m$, $B_j \in F_{v_j v_j}$, $1 \leq j \leq n$, $T_{i,r+j} \in F_{u_i v_j}$, $1 \leq i \leq m$, $1 \leq j \leq n$.

Then

$$\left[\begin{array}{ccc|ccc} A_1 & & & T_{1,m+1} & \cdots & T_{1,m+n} \\ & \ddots & & \vdots & & \vdots \\ & & A_m & T_{m,m+1} & \cdots & T_{m,m+n} \\ \hline & & & B_1 & & \\ & 0 & & & \ddots & \\ & & & & & B_n \end{array}\right] \tilde{S} \left[\begin{array}{ccc|ccc} A_1 & & & & & \\ & \ddots & & & 0 & \\ & & A_m & & & \\ \hline & & & B_1 & & \\ & 0 & & & \ddots & \\ & & & & & B_n \end{array}\right]$$

$\Rightarrow$ for $1 \leq i \leq m$, $1 \leq j \leq n$,

$$\begin{bmatrix} A_i & T_{i,m+j} \\ 0 & B_j \end{bmatrix} \tilde{S} \begin{bmatrix} A_i & 0 \\ 0 & B_j \end{bmatrix}.$$

PROOF: Let

$$M = \left[\begin{array}{ccc|ccc} \lambda I - A_1 & & & -T_{1,m+1} & \cdots & -T_{1,m+n} \\ & \ddots & & \vdots & & \vdots \\ & & \lambda I - A_m & -T_{m,m+1} & \cdots & -T_{m,m+n} \\ \hline & & & \lambda I - B_1 & & \\ & 0 & & & \ddots & \\ & & & & & \lambda I - B_n \end{array}\right]$$

and $D = \operatorname{diag}[\lambda I - A_1, \ldots, \lambda I - A_m, \lambda I - B_1, \ldots, \lambda I - B_n]$, so that M and D are matrices over the principal ideal domain $F[\lambda]$. The hypotheses, together with the fundamental theorem on similarity over a field, imply that M and D are equivalent over $F[\lambda]$.

Now let M_1 be obtained from M by replacing $T_{1,m+1}, \ldots, T_{1,m+n}$ with blocks of 0's. Note that to obtain a minor of M_1 with nonzero determinant, it is necessary that the number of rows deleted which pass through the block $\lambda I - A_1$, equal the number of columns deleted which pass through this block. It follows from this that every determinantal minor of M_1 is a determinantal minor of M also. Since $M \tilde{E} D$, we obtain that $M_1 \tilde{E} D$ as well. Write M_1 as $(\lambda I - A_1) \dot{+} M_2$ and D as $(\lambda I - A_1) + D_2$. It then follows from [5, Ch. 2, ex. 1] that $M_2 \tilde{E} D_2$. Repeating this process m times, we obtain that $M_m \tilde{E} D_m$, where

$$M_m = \left[\begin{array}{c|ccc} \lambda I - A_m & -T_{m,m+1} & \cdots & -T_{m,m+n} \\ \hline & \lambda I - B_1 & & \\ 0 & & \ddots & \\ & & & \lambda I - B_n \end{array}\right]$$

and $D = \operatorname{diag}[\lambda I - A_m, \lambda I - B_1, \ldots, \lambda I - B_n]$.

Arguing analogously on the columns of M_m, we obtain finally that

$$\left[\begin{array}{c|c} \lambda I - A_m & -T_{m,m+1} \\ \hline 0 & \lambda I - B_1 \end{array}\right] \tilde{E} \left[\begin{array}{c|c} \lambda I - A_m & 0 \\ \hline 0 & \lambda I - B_1 \end{array}\right]$$

from which it follows that

$$\left[\begin{array}{c|c} A_m & T_{m,m+1} \\ \hline 0 & B_1 \end{array}\right] \tilde{S} \left[\begin{array}{c|c} A_m & 0 \\ \hline 0 & B_1 \end{array}\right].$$

This establishes the lemma in the case when $i = m$ and $j = 1$.

To prove the lemma for arbitrary $(i, j) \in [1, m] \times [1, n]$, note that by simultaneous row and column permutation we may obtain

$$\left[\begin{array}{ccc|ccc} A_1 & & & & & \\ & \ddots & & & & \\ & & A_m & & & \\ & & & A_i & T_{i,m+j} & \cdots \\ \hline & & & & B_j & \\ & & & & & B_1 \\ & 0 & & & & \ddots \\ & & & & & \hat{B}_j \\ & & & & & \ddots \\ & & & & & B_n \end{array}\right] \tilde{S} \left[\begin{array}{ccc|ccc} A_1 & & & & & \\ & \ddots & & & & \\ & & A_m & & & \\ & & & A_i & & 0 \\ \hline & & & & B_j & \\ & & & & & B_1 \\ & 0 & & & & \ddots \\ & & & & & \hat{B}_j \\ & & & & & \ddots \\ & & & & & B_n \end{array}\right]$$

Running the above argument on this new pair of matrices, we get finally that

$$\left[\begin{array}{c|c} A_i & T_{i,m+j} \\ \hline 0 & B_j \end{array}\right] \tilde{S} \left[\begin{array}{c|c} A_i & 0 \\ \hline 0 & B_j \end{array}\right].$$

Q.E.D.

LEMMA 2: *Suppose* $A \in F_{rr}$, $B \in F_{ss}$, *and* $T \in F_{rs}$, *where both* A *and* B *are nonderogatory and in rational canonical form. Then (a)* $\exists\, \bar{X}, \bar{T} \in F_{rs}$ *such that* $\bar{T}$ *has nonzero entries only in its first column and* $T - \bar{T} = A\bar{X} - \bar{X}B$. *Also,* $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \tilde{S} \begin{bmatrix} A & \bar{T} \\ 0 & B \end{bmatrix}$. *(b)* $\exists\, \bar{\bar{X}}, \bar{\bar{T}} \in F_{rs}$ *such that* $\bar{\bar{T}}$ *has nonzero entries only in its last row and* $T - \bar{\bar{T}} = A\bar{\bar{X}} - \bar{\bar{X}}B$. *Also,* $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \tilde{S} \begin{bmatrix} A & \bar{\bar{T}} \\ 0 & B \end{bmatrix}$.

PROOF: By assumption,

$$A = \begin{bmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -\gamma_0 & -\lambda_1 & & \cdots & -\gamma_{r-1} \end{bmatrix} = C(f_A(x))$$

and

$$B = \begin{bmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -\eta_0 & -\eta_1 & & & -\eta_{s-1} \end{bmatrix} = C(f_B(x)),$$

where

$$f_A(x) = \gamma_0 + \gamma_1 x + \cdots + \gamma_{r-1}x^{r-1} + x^r$$

and

$$f_B(x) = \eta_0 + \eta_1 x + \cdots + \eta_{s-1}x^{s-1} + x^s.$$

Write $T = (t_{i,r+j})_{\substack{1 \le i \le r \\ 1 \le j \le s}}$.

(a) Perform the following $s-1$ sequences of elementary row and column operations on $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix}$, obtaining a sequence of matrices

$$\left\{ \begin{bmatrix} A & T_j \\ 0 & B \end{bmatrix} \right\}_{j=1}^{s-1}:$$

Sequence 1: For $1 \le i \le r$,

$$R_i \to R_i - t_{i,r+s} R_{r+s-1}$$

$$C_{r+s-1} \to C_{r+s-1} + t_{i,r+s} C_i.$$

These operations are effected by the similarity

$$\begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \to \begin{bmatrix} I_r & X_1 \\ 0 & I_s \end{bmatrix} \quad \begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \quad \begin{bmatrix} I_r & -X_1 \\ 0 & I_s \end{bmatrix} \equiv \begin{bmatrix} A & T_1 \\ 0 & B \end{bmatrix},$$

for an appropriate $X_1 \in F_{rs}$. Note that the last column of $T_1 \equiv (t^1_{k,r+l})_{\substack{1 \le k \le r \\ 1 \le l \le s}}$ consists entirely of 0's.

Sequence j ($2 \leq j \leq s-1$): For $1 \leq i \leq r$,

$$R_i \to R_i - t^{j-1}_{i,r+s-(j-1)} R_{r+s-j}$$

$$C_{r+s-j} \to C_{r+s-j} + t^{j-1}_{i,r+s-(j-1)} C_i.$$

These operations are effected by the similarity

$$\begin{bmatrix} A & T_{j-1} \\ 0 & B \end{bmatrix} \to \begin{bmatrix} I_r & X_j \\ 0 & I_s \end{bmatrix} \quad \begin{bmatrix} A & T_{j-1} \\ 0 & B \end{bmatrix} \quad \begin{bmatrix} I_r & -X_j \\ 0 & I_s \end{bmatrix} \equiv \begin{bmatrix} A & T_j \\ 0 & B \end{bmatrix},$$

for an appropriate $X_j \epsilon F_{rs}$. Note that the last j columns of $T_j \equiv (t^j_{k,r+l})_{\substack{1 \leq k \leq r \\ 1 \leq l \leq s}}$ consist entirely of 0's.

Now let $\bar{X} = \sum_{j=1}^{s-1} X_j$ and let $\bar{T} = T_{s-1}$. Then $\bar{T}$ has nonzero entries only in its first column and

$$\begin{bmatrix} I_r & \bar{X} \\ 0 & I_s \end{bmatrix} \quad \begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \quad \begin{bmatrix} I_r & -\bar{X} \\ 0 & I_s \end{bmatrix}$$

$$= \left\{ \prod_{j=1}^{s-1} \begin{bmatrix} I_r & X_{s-j} \\ 0 & I_s \end{bmatrix} \right\} \begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \left\{ \prod_{j=1}^{s-1} \begin{bmatrix} I_r & -X_j \\ 0 & I_s \end{bmatrix} \right\} = \begin{bmatrix} A & \bar{T} \\ 0 & B \end{bmatrix}$$

It follows from this that $T - \bar{T} = A\bar{X} - \bar{X}B$, also that

$$\begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \tilde{S} \begin{bmatrix} A & \bar{T} \\ 0 & B \end{bmatrix}.$$

(b) Perform the following $r-1$ sequences of elementary row and column operations on $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix}$, obtaining a sequence of matrices $\left\{ \begin{bmatrix} A & U_i \\ 0 & B \end{bmatrix} \right\}_{i=1}^{r-1}$:

Sequence 1. For $1 \leq j \leq s$,

$$C_{r+j} \to C_{r+j} - t_{1,r+j} C_2$$

$$R_2 \to R_2 + t_{1,r+j} R_{r+j}.$$

These operations are effected by the similarity

$$\begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \to \begin{bmatrix} I_r & Y_1 \\ 0 & I_s \end{bmatrix} \quad \begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \quad \begin{bmatrix} I_r & -Y_1 \\ 0 & I_s \end{bmatrix} \equiv \begin{bmatrix} A & U_1 \\ 0 & B \end{bmatrix},$$

for an appropriate $Y_1 \epsilon F_{rs}$. Note that the first row of $U_1 \equiv (u^1_{k,r+1})_{\substack{1 \leq k \leq r \\ 1 \leq l \leq s}}$ consists entirely of 0's.

Sequence i ($2 \leq i \leq r-1$). For $1 \leq j \leq s$,

$$C_{r+j} \to C_{r+j} - u^{i-1}_{i,r+j} C_{i+1}$$

$$R_{i+1} \to R_{i+1} + u^{i-1}_{i,r+j} R_{r+j}.$$

These operations are effected by the similarity

$$\begin{bmatrix} A & U_{i-1} \\ 0 & B \end{bmatrix} \to \begin{bmatrix} I_r & Y_i \\ 0 & I_s \end{bmatrix} \quad \begin{bmatrix} A & U_{i-1} \\ 0 & B \end{bmatrix} \quad \begin{bmatrix} I_r & -Y_i \\ 0 & I_s \end{bmatrix} \equiv \begin{bmatrix} A & U_i \\ 0 & B \end{bmatrix},$$

for an appropriate $Y_i \epsilon F_{rs}$. Note that the first i rows of $U_i \equiv (u^i_{k,r+l})_{\substack{1 \le k \le r \\ 1 \le l \le s.}}$ consist entirely of 0's.

Now let $\bar{\bar{X}} = \sum_{i=1}^{r-1} Y_i$ and let $\bar{\bar{T}} = U_{r-1}$. Then $\bar{\bar{T}}$ has nonzero entries only in its last row and

$$\begin{bmatrix} I_r & \bar{\bar{X}} \\ 0 & I_s \end{bmatrix} \quad \begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \quad \begin{bmatrix} I_r & -\bar{\bar{X}} \\ 0 & I_s \end{bmatrix} = \left\{ \prod_{i=1}^{r-1} \begin{bmatrix} I_r & U_{r-i} \\ 0 & I_s \end{bmatrix} \right\} \begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \left\{ \prod_{i=1}^{r-1} \begin{bmatrix} I_r & -U_i \\ 0 & I_s \end{bmatrix} \right\} = \begin{bmatrix} A & \bar{\bar{T}} \\ 0 & B \end{bmatrix}.$$

It follows from this that $T - \bar{\bar{T}} = A\bar{\bar{X}} - \bar{\bar{X}}\, B$, also, that

$$\begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \tilde{S} \begin{bmatrix} A & \bar{\bar{T}} \\ 0 & B \end{bmatrix}. \qquad \text{Q.E.D.}$$

LEMMA 3: *Suppose* $A \epsilon F_{rr}$, $B \epsilon F_{ss}$, *and* $T \epsilon F_{rs}$, *where both* A *and* B *are nonderogatory and in rational canonical form. Assume also that* f_A *and* f_B *are both powers of the same monic irreducible polynomial* p(x) *and that* $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \tilde{S} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. *Then* T *must be* 0 *if*

(*a*) $r \le s$ *and* T *has nonzero entries possibly only in its first column*

or

(*b*) $s \le r$ *and* T *has nonzero entries possibly only in its last row.*

PROOF: By hypothesis,

$$A = \begin{bmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & 0 & 0 \ldots & & 1 \\ -\gamma_0 & -\gamma_1 & & & -\gamma_{r-1} \end{bmatrix} = C(f_A)$$

and

$$B = \begin{bmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & 0 & 0 \ldots & & 1 \\ -\eta_0 & -\eta_1 & \ldots & & -\eta_{s-1} \end{bmatrix} = C(f_B)$$

where

$$f_A(x) = \gamma_0 + \gamma_1 x + \ldots + \gamma_{r-1}x^{r-1} + x^r = p(x)^e$$

and

$$f_B(x) = \eta_0 + \eta_1 x + \ldots + \eta_{s-1}x^{s-1} + x^s = p(x)^f,$$

for some integers e and f. Write $T = (t_{i,r+j})_{\substack{1 \le i \le r \\ 1 \le j \le s}}$.

Now let $D=\text{diag}[\lambda I-A,\ \lambda I-B]\in F[\lambda]_{(r+s),(r+s)}$. For $(D)_{ij}$ any $(r+s-1)\times(r+s-1)$ minor of D, it may be seen that $\det(D)_{ij}\neq 0\Rightarrow i,j\leq r$ or $i,j\geq r+1$. Note also that for $i,j\leq r$, $\det(D)_{ij}=g_{ij}(\lambda)f_B(\lambda)$, for some $g_{ij}(\lambda)\in F[\lambda]$, where $gr_i(\lambda)=\pm 1$; and that for $i,j\geq r+1$, $\det(D)_{ij}=h_{ij}(\lambda)f_A(\lambda)$, for some $h_{ij}(\lambda)\in F[\lambda]$, where $h_{r+s,\,r+1}(\lambda)=\pm 1$.

Let Δ_{r+s-1} be the $(r+s-1)\times(r+s-1)$determinantal divisor of D. It then follows from the above calculations that

$$\Delta_{r+s-1}=g.c.d.\ \{f_A(\lambda),f_B(\lambda)\}=\begin{cases}f_A(\lambda), & \text{if } r\leq s\\ f_B(\lambda), & \text{if } s\leq r.\end{cases}$$

Note also that $\begin{bmatrix}A & T\\ 0 & B\end{bmatrix}\tilde{S}\begin{bmatrix}A & 0\\ 0 & B\end{bmatrix}\Rightarrow\begin{bmatrix}\lambda I-A & -T\\ 0 & \lambda I-B\end{bmatrix}$ is equivalent over $F[\lambda]$ to D. Hence for all $i,j\geq r+s$,

$$\Delta_{r+s-1}\,|\det\left(\begin{bmatrix}\lambda I-A & -T\\ 0 & \lambda I-B\end{bmatrix}_{ij}\right).$$

We now prove (a) and (b).

(a) It may be seen that for $i\leq r$, $\det\left(\begin{bmatrix}\lambda I-A & -T\\ 0 & \lambda I-B\end{bmatrix}\right)_{r+s,i}=\pm t_{i,r+1}\ \lambda^{r-1}+q(\lambda)+E$, where $q(\lambda)$ is a polynomial of degree $\leq r-2$ and E is an $(r+s-1)\times(r+s-1)$ determinantal minor of D. It follows that

$$\Delta_{r+s-1}|\pm t_{i,r+1}\ \lambda^{r-1}+q(\lambda)+E.$$

Since $\Delta_{r+s-1}|E$ and since $\Delta_{r+s-1}=f_A(\lambda)$ is a polynomial of degree r in this case, we obtain that $t_{i,r+1}=0$, $i\leq r$, whence $T=0$.

(b) It may be seen that $\det\left(\begin{bmatrix}\lambda I-A & -T\\ 0 & \lambda I-B\end{bmatrix}\right)_{r+s,1}=\pm t_{r,r+1}\pm t_{r,r+2}\ \lambda\pm t_{r,r+3}\ \lambda^2\pm\ \ldots\ \pm t_{r,r+s}\ \lambda^{s-1}+F$, where F is an $(r+s-1)\times(r+s-1)$ determinantal minor of D. It follows that

$$\Delta_{r+s-1}|\pm t_{r,r+1}\pm t_{r,r+2}\ \lambda\pm\ \ldots\ \pm t_{r,r+s}\ \lambda^{s-1}+F.$$

Since $\Delta_{r+s-1}|F$ and since $\Delta_{r+s-1}=f_B(\lambda)$ is a polynomial of degree s in this case, we obtain that $t_{r,r+1}=t_{r,r+2}=\ \ldots\ =t_{r,r+s}=0$, whence $T=0$. Q.E.D.

It is now convenient to prove our main result in a simple special case.

Lemma 4: Suppose $A\in F_{rr}$, $B\in F_{ss}$, *and* $T\in F_{rs}$, *where both* A *and* B *are nonderogatory. Assume also that* $f_A(x)=p_1(x)^d$ *and* $f_B(x)=p_2|x(^e$, *where* $p_1(x)$ *and* $p_2(x)$ *are monic irreducible polynomials.*

Then $\begin{bmatrix}A & T\\ 0 & B\end{bmatrix}\tilde{S}\begin{bmatrix}A & 0\\ 0 & B\end{bmatrix}\Rightarrow T=AX-XB$, *for some* $X\in F_{rs}$.

PROOF: As noted above, we may assume w.l.o.g. that $A=RF(A)$ and $B=RF(B)$. If $p_1(x)\neq p_2(x)$, then A and B have no eigenvalues in common, and hence we know from Lemma 0 that $\exists X\in F_{rs}$ such that $AX-XB=T$ (the hypothesis on similarity is superfluous in this case.)

Assume now that $p_1(x)=p_2(x)$. If $r\leq s$, use Lemma 2a to find $\bar{X},\bar{T}\in F_{rs}$ such that $\bar{T}$ has nonzero entries only in its first column, $T-\bar{T}=A\bar{X}-\bar{X}B$, and $\begin{bmatrix}A & T\\ 0 & B\end{bmatrix}\tilde{S}\begin{bmatrix}A & T\\ 0 & B\end{bmatrix}$. Since $\begin{bmatrix}A & T\\ 0 & B\end{bmatrix}\tilde{S}\begin{bmatrix}A & 0\\ 0 & B\end{bmatrix}$,

it follows that $\begin{bmatrix} A & \bar{T} \\ 0 & B \end{bmatrix} \tilde{S} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, and we then obtain from Lemma 3a that $\bar{T}=0$. Thus $T=A\bar{X}-\bar{X}B$ in this case. If $s \leq r$, use Lemma 2b to find $\bar{\bar{X}}, \bar{\bar{T}} \in F_{rs}$ such that $\bar{\bar{T}}$ has nonzero entries only in its last row, $T-\bar{\bar{T}}=A\bar{\bar{X}}-\bar{\bar{X}}B$, and $\begin{bmatrix} A & \bar{\bar{T}} \\ 0 & B \end{bmatrix} \tilde{S} \begin{bmatrix} A & T \\ 0 & B \end{bmatrix}$. Again, $\begin{bmatrix} A & \bar{\bar{T}} \\ 0 & B \end{bmatrix} \tilde{S} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, and we then obtain from Lemma 3b that $\bar{\bar{T}}=0$. Thus $T=A\bar{\bar{X}}-\bar{\bar{X}}B$ in this case as well.

Q.E.D.

We now drop the requirement that A and B be nonderogatory.

LEMMA 5: *Suppose* $A \in F_{rr}$, $B \in F_{ss}$, *and* $T \in F_{rs}$. *Assume that both* A *and* B *are nonzero and that* $f_A(x)=p_1(x)^d$ *and* $f_B(x)=p_2(x)^e$, *where* $p_1(x)$ *and* $p_2(x)$ *are monic irreducible polynomials. Then* $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \tilde{S} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \Rightarrow T=AX-XB$ *for some* $X \in F_{rs}$.

PROOF: As before, we may assume that $A=RF(A)$, $B=RF(B)$, and that $p_1(x)=p_2(x)$. We then have that $A=\text{diag}[C(p(x)^{d_1}), C(p(x)^{d_2}), \ldots, C(p(x)^{d_u})]$, where $d=d_1 \geq d_2 \geq \ldots \geq d_u$, and $B=\text{diag}[C(p(x)^{e_1}), C(p(x)^{e_2}), \ldots, C(p(x)^{e_v})]$, where $e=e_1 \geq e_2 \geq \ldots \geq e_v$. Now write $T=(T_{i,u+j})_{\substack{1 \leq i \leq u \\ 1 \leq j \leq v}}$, where $T_{i,u+j}$ has d_i rows and e_j columns. We then have by Lemma 1 that for all $(i, j) \in [1, u] \times [1, v]$,

$$\begin{bmatrix} C(p(x)^{d_i}) & T_{i,u+j} \\ 0 & C(p(x)^{e_j}) \end{bmatrix}$$

$$\tilde{S} \begin{bmatrix} C(p(x)^{d_i}) & 0 \\ 0 & C(p(x)^{e_j}) \end{bmatrix}.$$

It then follows from Lemma 4 that there is a matrix $X_{i,u+j}$ over F such that

$$T_{i,u+j}=C(p(x)^{d_i})\, X_{i,u+j}-X_{i,u+j}\, C(p(x)^{e_j}).$$

Let $X=(X_{i,u+j})_{\substack{1 \leq i \leq u \\ 1 \leq j \leq v}} \in F_{rs}$. We then obtain by straightforward computation that $T=AX-XB$.

Q.E.D.

We now establish the main result.

THEOREM 6: *Suppose* $A \in F_{rr}$, $B \in F_{ss}$, *and* $T \in F_{rs}$. *Then* $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \tilde{S} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \Rightarrow T=AX-XB$, *for some* $X \in F_{rs}$.

PROOF: As before, we may assume that both A and B are nonzero and in rational canonical form. Assume also that

$$f_A(x)=p_1(x)^{d_1}p_2(x)^{d_2} \ldots p_u(x)^{d_u}$$

and

$$f_B(x)=q_1(x)^{e_1}q_2(x)^{e_2} \ldots q_v(x)^{e_v},$$

where $\{p_i(x)\}_{i=1}^{u}$ and $\{q_j(x)\}_{j=1}^{v}$ are sets of distinct irreducible polynomials in $F[x]$ We may then write $A=\text{diag}[G_1, \ldots, G_u]$ and $B=\text{diag}[H_1, \ldots, H_v]$, where $f_{G_i}(x)=p_1(x)^{d_i}$ and $f_{H_j}(x)=q_j(x)^{e_j}$. Now write $T=(T_{i,u+j})_{\substack{1 \leq i \leq u \\ 1 \leq j \leq v}}$, where $T_{i,u+j}$ is conformable with G_i and H_j. We then have by Lemma 1 that for all $(i, j) \in [1, u] \times [1, v]$, $\begin{bmatrix} G_i & T_{i,u+j} \\ 0 & H_j \end{bmatrix} \tilde{S} \begin{bmatrix} G_i & 0 \\ 0 & H_j \end{bmatrix}$. It then follows from Lemma 5 that there is a matrix $X_{i,u+j}$ over F such that $T_{i,u+j}=G_iX_{i,u+j}-X_{i,u+j}H_j$. Let $X=(X_{i,u+j})_{\substack{1 \leq i \leq u \\ 1 \leq j \leq v}} \in F_{rs}$. We then obtain by straightforward computation that $T=AX-XB$. Q.E.D.

COROLLARY 6.1: *Suppose* $A \epsilon F_{rr}$, $B \epsilon F_{ss}$, *and* $T, \tilde{T} \epsilon F_{rs}$. *Then*

$$\begin{bmatrix} A & T-\tilde{T} \\ 0 & B \end{bmatrix} \tilde{S} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \Rightarrow \begin{bmatrix} A & \tilde{T} \\ 0 & B \end{bmatrix} \tilde{S} \begin{bmatrix} A & \tilde{T} \\ 0 & B \end{bmatrix}.$$

PROOF: The hypotheses, together with Theorem 6, imply $T-\tilde{T}=AX-XB$, for some $X \epsilon F_{rs}$. It is then a simple computation that

$$\begin{bmatrix} I_r & -X \\ 0 & I_s \end{bmatrix} \begin{bmatrix} A & \tilde{T} \\ 0 & B \end{bmatrix} \begin{bmatrix} I_r & X \\ 0 & I_s \end{bmatrix} = \begin{bmatrix} A & T \\ 0 & B \end{bmatrix},$$

so that

$$\begin{bmatrix} A & \tilde{T} \\ 0 & B \end{bmatrix} \tilde{S} \begin{bmatrix} A & T \\ 0 & B \end{bmatrix}.$$

Q.E.D.

Note that the converse to Corollary 6.1 fails. For example, let $F=$ the reals, R, $A=B=(3)$, $T=(4)$, and $\tilde{T}=(2)$. Then $\begin{bmatrix} 3 & 4 \\ 0 & 3 \end{bmatrix} \tilde{S} \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$, since $\begin{bmatrix} 3-x & 4 \\ 0 & 3-x \end{bmatrix}$ is equivalent over $R[x]$ to $\begin{bmatrix} 3-x & 2 \\ 0 & 2 \end{bmatrix}$, but $\begin{bmatrix} 3 & 4-2 \\ 0 & 3 \end{bmatrix}$ is *not* similar to $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, since there is obviously no x satisfying $4-2=x(3-3)$.

We note two further consequences of the results and techniques developed thus far. First, they may be used to prove the converse to Lemma 0, namely that if $A \epsilon F_{rr}$ and $B \epsilon F_{ss}$ have the property that for all $T \epsilon F_{rs}$ there is an $X \epsilon F_{rs}$ such that $T=AX-XB$, then A and B have no eigenvalues in common. Second, they may be used to find an explicit solution in X of the matric equation $T=AX-XB$, at least in the case when A and B are in rational cononical form. See [1] for another approach to solving this equation.

We conclude with a simple generalization of Theorem 6.

THEOREM 7: *Suppose* $U_i \epsilon F_{r_i r_i}$, $1 \leq i \leq k$, *and* $N_{ij} \epsilon F_{r_i r_j}$, $1 \leq i < j \leq k$. *Then*

$$\begin{bmatrix} U_1 & & & \\ & \ddots & & \\ & & N_{ij} & \\ & & & U_k \end{bmatrix} \tilde{S} \begin{bmatrix} U_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & U_k \end{bmatrix} \Longleftrightarrow$$

for each $i, j \leq k$ $\exists X_{ij} \epsilon F_{r_i r_j}$ *such that* $U_i X_{ij} - X_{ij} U_j = N_{ij}$.

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On the Dimension Group of Classical Physics

Chester H. Page

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234
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The basic principles of the group properties of physical quantities are reviewed. The problems associated with the dimensions of *angle* and *logarithm* are solved by using functional equations, instead of analytic expressions, for defining functions of nonnumerical quantities. It is concluded that the neper and radian are related by $\mathrm{Np} = -\mathrm{j}$ rad, so that when the radian is considered as a base unit, the neper becomes a derived unit, and is the SI unit for logarithmic quantities.

Key words: Angle; dimension; logarithm; neper; radian.

1. Introduction

Theoretical physics and engineering are based on mathematical models of the phenomena of nature, i.e., the relations among measurable physical entities are expressed by mathematical equations. Scientists originally considered these equations to be relations among *numbers*, with the numbers representing the measures of quantities expressed in terms of agreed-upon units. Maxwell was one of the first to realize that this interpretation of equations in science is sterile compared to the *quantity calculus* concept. In the modern *quantity equation* viewpoint, the letter symbols in an equation represent mathematical elements, called *quantities*, which are in a one-to-one correspondence with the measurable entities of the laboratory.

In measurement, the physical entity corresponding to the mathematical element (quantity) q is measured by its ratio to a standard entity sample called a *measurement unit*. The mathematical counterpart of this measurement unit is the *symbolic unit*, $[q]$. The result of a measurement is thus modeled by the relation

$$q = \{q\}[q] \tag{1}$$

expressing the *values* of the quantity as the product of a positive real number, $\{q\}$ (the *measure* or *numerical value* of the quantity), and the chosen unit. Since the laws of physics are observed as proportionalities among measures, these laws are modeled by multiplicative relations.

If, experimentally, the measure of a quantity q is proportional to the product of the measures of the quantities p and s, the experimenter's measure equation

$$\{q\} = k\{p\}\{s\} \tag{2}$$

can be generalized to the theoretical quantity equation

$$q = n\,p\,s \tag{3}$$

where q, p, s are mathematical elements in the *dimension* spaces $D(q)$, $D(p)$, $D(s)$, and n is a number. $D(q)$ is commonly called "the dimension of q". By taking the dimension space of number, $D(n)$, as the identity element, the set of dimensions becomes an Abelian group. The "product"

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$p\ s$ is not yet defined. However, a postulate on the completeness of a physical law (see reference [1][1]) implies that the product operation has the following property:

"The product of *any* point in $D(p)$ and *any* point in $D(s)$ is a point in $D(q)$, and conversely, any point in $D(q)$ can be expressed as the product of some point in $D(p)$ and some point in $D(s)$."

It follows that

$$D(ps) = D(p)\ D(s)$$

The quantity equation can be written

$$\{q\}\,[q] = n\{p\}\{s\}[p][s] \tag{4}$$

If the units of q, p, and s are chosen to make $n = k$, then

$$[q] = [p]\ [s] \tag{5}$$

and the system of units is said to be *coherent* (strictly speaking, coherent with the system of quantities and equations used to represent the theory).

The number of independent dimensions is a matter of taste and convenience. In the International System of Units, those "units which by convention are regarded as dimensionally independent" [2] are called "base units."

2. Supplementary Units

The dimension space of "angle" is frequently taken as identical to the dimension space of numbers; the quantity, angle, is then numeric, and its unit, radian, can be interpreted as a special name for the number 1. On the other hand, it is often convenient to distinguish "work" and "torque" as being entities of different kinds, by assigning them to different dimension spaces. In this case, angles are not numbers, and the radian is a base unit. The General Conference on Weights and Measures has very properly declined to make a dogmatic decision on this question of taste and convenience, so has classified the radian (and steradian) as *supplementary units* in the SI.

The International Organization for Standardization (ISO), although it no longer recognizes the use of radian as a base unit, formerly gave two definitions of "angle." One of these definitions was based on the concept of "analytical angle," defined as the ratio of the arc cut out on a circle to the radius of that circle. This concept makes "angle" a pure number. The second definition was based on the concept of "geometrical angle," which expresses the degree of divergence between two intersecting straight lines. In this concept, which is usually used in technology, "angle" possesses its own nature and dimension, and the radian is a base unit.

The engineering practice of making a dimensional distinction between "number" and "angle" is related to the existence, and use, of a multiplicity of units. An angle cannot be completely described by a number alone: the unit (radian, degree, grade, etc.) must also be stated. It is therefore meaningless to say, for example, that a certain angle is greater than 2.

3. Logarithmic Quantities

The nature of the units and dimensions of "logarithmic quantities" such as *attenuation* and *level* raises problems similar to those raised by "angle." The attenuation of a "signal" (communication signal, light beam, electric power, etc.) can be measured by comparison with the attenuation associated with a standard reference situation.

[1] Figures in brackets indicate the literature references at the end of the paper.

A 1-bel standard for attenuation in a transmission path makes the output (transmitted) power 1/10 of the input (incident) power and makes the output voltage and current $\sqrt{1/10}$ of the input voltage and current. A 1-neper standard makes the output power $1/e^2$ of the input power, and the output voltage and current 1/e of the input voltage and current. Thus a given attenuation can be expressed as, say, 1 bel (commonly given as 10 decibels) or as 1.151 nepers. The given quantity, attenuation, is expressed as the product of a numerical value and a unit, and the prevalence of two different units makes such a statement as "The attenuation was 3" an incomplete statement. The quantity, attenuation, is therefore not a pure number, but is an element in some other dimension space.

Since attenuation can be expressed as the logarithm of the ratio of two values of a quantity, it is the logarithm of a number. This creates the paradoxical situation wherein a nonnumerical dimension is assigned to a function of a pure number.

The major object of this paper is to resolve this paradox, i.e., to explore the existence of possible pairings of numeric and nonnumeric functions of numeric and nonnumeric variables.

4. Structure of the Group of Quantities

A set of coherent units forms a group which is isomorphic to the group of dimensions. Since the primitive quantities can be expressed as products of units by positive real numbers, they constitute the direct product group $\mathscr{P} \times \mathscr{G}$, where $\mathscr{P}$ is the group of positive real numbers, and $\mathscr{G}$ the Abelian multiplicative group of units. For mathematical treatment of quantities, it is necessary to choose reference directions or polarities, creating negative measures. The quantity group becomes $\mathscr{R} \times \mathscr{G}$, where $\mathscr{R}$ is the field of real numbers.

The group $\mathscr{R} \times \mathscr{G}$ has the following properties:

If au denotes the element in $\mathscr{R} \times \mathscr{G}$ determined by the element a of $\mathscr{R}$ and the element u of $\mathscr{G}$, then

$$(a+b)\,u = au + bu$$

and

$$(au)\,(bv) = (ab)uv \tag{6}$$

hold for any a, b in $\mathscr{R}$ and any u, v in $\mathscr{G}$.
Relations like

$$au + bv = cu + dv \tag{7}$$

can always be decomposed into independent relations:

$$\begin{aligned} au = cu &\Rightarrow a = c \\ bv = dv &\Rightarrow b = d \end{aligned} \tag{8}$$

and the equations of physics can be written in dimensionally homogeneous form.

In physics and engineering, it is frequently convenient, for mathematical manipulation, to use complex numbers. Basically, this is an artifice to allow the simultaneous treatment of pairs of complementary quantities. The familiar elementary example is the simultaneous treatment of sine and cosine components of an oscillatory solution of a differential equation. This is a special case of a general artifice using hypercomplex quantities to simultaneously manipulate related quantities having different dimensions [3]. The usual occurrence of this device is in the form

$$au + \mathrm{j}bv$$

where a and b are real numbers, and the unit imaginary, j, serves as a redundant but useful "flag."

We first examine the implications of allowing angle to be nonnumeric. If we define cos θ and sin θ, as *quantities*, by

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$$\cos\theta=\sum_{n=0}^{\infty}\frac{(-1)^n\theta^{2n}}{2n!}$$

$$\sin\theta=\sum_{n=0}^{\infty}\frac{(-1)^n\theta^{2n+1}}{(2n+1)!} \tag{9}$$

then each term of the expansion on the right must have the same dimension, and this must be the dimension of the quantity on the left. This "dimensional homogeneity" requires

$$D^2(\theta)=D(\theta)D(\theta)=D(1)$$

$$D(\cos\theta)=D(1) \tag{10}$$

$$D(\sin\theta)=D(\theta)$$

If we interpret eq (9) to be only numerical equations with θ representing the measure of the angle relative to the radian as a unit, no such conclusion can be drawn. In that case, we must consider some other definition of angle as a *quantity*, which is compatible with (9) as a relation among numerical values.

Now cos θ and sin θ can also be defined as appropriate solutions of the differential relations

$$\mathrm{d}\sin\theta=\cos\theta\,\mathrm{d}\theta$$

$$\mathrm{d}\cos\theta=-\sin\theta\,\mathrm{d}\theta \tag{11}$$

In this case, dimensional homogeneity requires

$$D(\sin\theta)=D(\cos\theta)\,D(\theta)$$

$$D(\cos\theta)=D(\sin\theta)\,D(\theta) \tag{12}$$

which again implies

$$D^2(\theta)=D(1)$$

Note, however, that if $D(\theta)\neq D(1)$, then $D(\sin\theta)\neq D(\cos\theta)$, and the dimensions of various functions of θ are not known *a priori;* in particular, functions of θ are not necessarily numerical.

5. Classes of Functions in $\mathscr{R}\times\mathscr{G}$

Denote the unity element of $\mathscr{G}$ by 1. Since 1 is idempotent, a function $f(x)$ defined on $\mathscr{R}$ can also be considered as defined on $\mathscr{R}\times 1$:

$$f(x)\epsilon\mathscr{R}\times 1,\quad x\epsilon\mathscr{R}\times 1 \qquad \text{CLASS (A)}$$

Generalization to $f(x)$ with $f\epsilon\mathscr{R}\times\mathscr{G}$ and $x\epsilon\mathscr{R}\times\mathscr{G}$ develops the additional classes of functions:

$$F(x)\epsilon\mathscr{R}\times\mathscr{G},\quad x\epsilon\mathscr{R}\times 1 \qquad \text{CLASS (B)}$$

$$f(X)\epsilon\mathscr{R}\times 1,\quad X\epsilon\mathscr{R}\times\mathscr{G} \qquad \text{CLASS (C)}$$

$$F(X)\epsilon\mathscr{R}\times\mathscr{G},\quad X\epsilon\mathscr{R}\times\mathscr{G} \qquad \text{CLASS (D)}$$

The use of lower-case letters for elements of $\mathscr{R}$ (or $\mathscr{R} \times 1$) and capital letters for general elements of $\mathscr{B} \times \mathscr{G}$ will simplify the presentation, and avoid the need for repeated statements about domains and ranges.

Since multiplication is defined in $\mathscr{R} \times \mathscr{G}$, multiplication of non-numerical quantities generates trivial examples of Class D. Generalization of other functions will be based on *implicit* definitions provided by functional equations, in particular, by addition theorems and multiplication theorems possessed on $\mathscr{R}$.

For classes C and D, multiplication theorems are inappropriate; addition theorems are required, with the terms of the argument containing the same factor from $\mathscr{G}$. For class B, a multiplication theorem is required to separate this class from D by restricting the domain.

The lack of addition on $\mathscr{G}$ necessitates dimensional homogeneity in equations; the general multiplication and addition theorems for classes (B), (C), and (D) are:

$$F(xy) = g(x, y)\, F(y) + g(y, x)\, F(x) \qquad \text{(B)} \quad (13)$$

$$f(X + Y) = h\{f(X), f(Y)\} + h\{f(Y), f(X)\} \qquad \text{(C)} \quad (14)$$

$$F(X + Y) = g(X, Y)\, F(Y) + g(Y, X)\, F(X) \qquad \text{(D)} \quad (15)$$

where

$$X = \{X\}\, U$$

$$Y = \{Y\}\, U$$

and $U \epsilon \mathscr{G}$.

Although F is a mapping of $\mathscr{R}$ onto $\mathscr{R} \times \mathscr{G}$, it is implicitly defined by a functional equation $E(F)$ in its own image space. The inverse mapping is similarly defined by an equation $e(f)$ in its image space. The mappings thus give

$$F : \mathscr{R} \to \mathscr{R} \times \mathscr{G}, \qquad e(f) \to E'(F)$$

$$f : \mathscr{R} \times \mathscr{G} \to \mathscr{R}, \qquad E(F) \to e'(f).$$

The relations so far are completely general, permitting an infinity of functions satisfying pairs of multiplication/addition theorems. If we wish to select a set of functions having special properties, the simplest and most interesting set comprises those functions having unique functional theorems, i.e., $E'(F) \equiv E(F), e'(f) \equiv e(f)$ so that

$$F : e(f) \to E(F)$$

$$f : E(F) \to e(f).$$

Since

$$F(f(X + Y)) \equiv X + Y = F(f(X)) + F(f(Y)), \qquad (16)$$

the inverse of (14) is

$$F(x) + F(y) = F\{h(x, y) + h(y, x)\} \qquad (17)$$

where

$$\mathrm{x} \equiv f(X)$$

$$X \equiv F(x).$$

For (17) and (13) to have identical form, (14) and (13) must be

$$f(X+Y)=f(X)f(Y) \tag{18a}$$

$$F(xy)=F(x)+F(y) \tag{18b}$$

The $\mathcal{G}$-component of F (the unit of F) can be factored out of (18b), making the equation *appear* to be a trivial combination of the solution of

$$f(xy)=f(x)+f(y) \tag{19}$$

and an arbitrary unit of $\mathcal{G}$. For *real* x, (19) has the well-known solution

$$f(x)=\log_a x \tag{20}$$

where the logarithm may be taken to any base.

Since $f(x)$ is the measure $\{F(x)\}$, its different numerical values associated with different logarithmic bases may be interpreted as different measures corresponding to different choices of the size of the unit of F, that is, to different choices of a when $F(a)$ is the sample taken as the unit of F.

Since the *quantity* $F(x)$ is invariant to unit choice, we can write

$$F(x)=\{F(x)\}_a F(a)=\{F(x)\}_b F(b) \tag{21}$$

where $\{F\}_a$ denotes the measure of F when $F(a)$ is the unit. In terms of $F(b)$ as the unit,

$$F(a)=\{F(a)\}_b F(b) \tag{22}$$

Combining (21) and (22) yields

$$\frac{\{F(x)\}_b}{\{F(x)\}_a}\equiv\{F(a)\}_b \tag{23}$$

as the relation between different measures of $F(x)$.

Now

$$\frac{\log_b x}{\log_a x}\equiv\log_b a \tag{24}$$

so that the solution (20) can be written

$$\{F(x)\}_a=\log_a x \tag{25}$$

implying that $F(x)$ is a generalized *nonnumerical* logarithm:

$$\begin{aligned} F(x)&=\mathrm{LOG}\ x \\ &=\log_a x\ \mathrm{LOG}\ a \end{aligned} \tag{26}$$

with LOG a an element of $\mathcal{G}$. The engineering usages of *neper* (Np) and *bel* as units are interpretable as usage of LOG e and LOG $\sqrt{10}$ as units, with the quantity, attenuation (gain, loss), defined by LOG $\sqrt{P_1/P_2}$, where P_1 and P_2 are the power at points 1 and 2 of a system:

$$\mathrm{LOG\ e}=\mathrm{Np} \tag{27}$$

With the introduction of the complex number artifice, (18b) has also the solution

$$F_2(x) = \mathrm{ARG}\ x \overset{d}{=} \mathrm{ARCTAN}\frac{\mathrm{Im}\ x}{\mathrm{Re}\ x} \tag{28}$$

For the inverse to be unique, the modulus of x must be fixed, conveniently at unity. The "natural" unit of F_2 is given by

$$\mathrm{ARG}\ e^{j} = \mathrm{rad} \tag{29}$$

Equations (18) have, therefore, two primitive solutions:

(1) For $x, y \in \mathscr{R}$,

$$F_1(x) = \mathrm{LOG}\, x$$

mapping the multiplicative group $\mathscr{R}$ onto an additive group $\mathscr{L}$ of logarithms:

$$F_1(\mathscr{R}) = \mathscr{L}$$

$$f_1(\mathscr{L}) = \mathscr{R} \tag{30}$$

(2) For $x, y \in \mathscr{C}_1$ (the multiplicative group of complex numbers of unit modulus)

$$F_2(x) = \mathrm{ARG}\ x \overset{d}{=} \mathrm{ARCTAN}\frac{\mathrm{Im}\ x}{\mathrm{Re}\ x}$$

$$F_2(\mathscr{C}_1) = \Theta \tag{31}$$

$$f_2(\Theta) = \mathscr{C}_1$$

where Θ is the additive group of angles.

These two primitive solutions can be combined into a general solution:

$$F(\mathscr{C}) = F(\mathscr{R} \times \mathscr{C}_1) = F(\mathscr{R}) \cup F(\mathscr{C}_1) = F_1(\mathscr{R}) \cup F_2(\mathscr{C}_1) = \mathscr{L} \cup \Theta \tag{32}$$

$$f(\mathscr{L} \cup \Theta) = f(\mathscr{L}) \times f(\Theta) = f_1(\mathscr{L}) \times f_2(\Theta) = \mathscr{R} \times \mathscr{C}_1 = \mathscr{C} \tag{33}$$

where $\mathscr{C}$ is the complex number field. Thus F maps the product of the two multiplicative groups $\mathscr{R}$ and $\mathscr{C}_1$ onto the union of the two additive groups $\mathscr{L}$ and Θ.

Equation (18a) implies

$$f(nX) = (f(X))^n \tag{34}$$

for integral n.

Since $f(X) \in \mathscr{C}$, the right hand side of (34) is defined for general complex numbers z in place of n. Generalization of (34) yields

$$f(zX) = (f(X))^z \tag{35}$$

Now (27) and (29) imply, respectively,

$$f(\mathrm{Np}) = \mathrm{e} \tag{36}$$

$$f(\mathrm{rad}) = e^{\mathrm{j}}$$

so that (35) yields

$$f(\mathrm{jNp}) = \mathrm{e}^{\mathrm{j}} = f(\mathrm{rad})$$

$$f(-\mathrm{j\ rad}) = \mathrm{e}^{1} = f(\mathrm{Np}) \tag{37}$$

or

$$\mathrm{Np} \equiv -\mathrm{j\ rad} \tag{38}$$

Thus when the radian is considered to be a base unit, the neper is dimensionally the same as a radian, if the imaginary j is considered dimensionally numeric. When *radian* is considered to be a special name for the number 1, *neper* becomes a special name for $-\mathrm{j}$, not a special name for the number 1 as is sometimes stated.

Equation (38) also implies that

$$\mathscr{L} \equiv \mathscr{R} \times [\mathrm{LOG}] = \mathrm{j}\mathscr{R} \times \Theta \tag{39}$$

and since

$$\Theta \equiv \mathscr{R} \times \Theta,$$

we have

$$\mathscr{L} \cup \Theta = \mathscr{C} \times \Theta \tag{40}$$

so that

$$F(\mathscr{C}) = \mathscr{C} \times \Theta$$

$$f(\mathscr{C} \times \Theta) = \mathscr{C}. \tag{41}$$

Thus F is a mapping of complex numbers onto complex angles.

In particular, measuring in radians and nepers,

$$\begin{aligned} f(\theta) = f(\{\theta\}\ \mathrm{rad}) = (f(\mathrm{rad}))^{\{\theta\}} = \mathrm{e}^{\mathrm{j}\{\theta\}} = \mathrm{e}^{\mathrm{j}\,\theta/\mathrm{rad}} \\ f(L) = f(\{L\}\ \mathrm{Np}) = \mathrm{e}^{\{L\}} = \mathrm{e}^{L/\mathrm{Np}} \end{aligned} \tag{42}$$

and for

$$z = |z|\ \mathrm{e}^{\mathrm{j}\{\theta\}},$$

$$\begin{aligned} F(z) &= \mathrm{LOG}\ |z| + \{\theta\}\ \mathrm{rad} \\ &= \ln\ |z|\ \mathrm{Np} + \{\theta\}\ \mathrm{jNp} \\ &= (\ln\ |z| + \mathrm{j}\ \{\theta\})\ \mathrm{Np}. \end{aligned} \tag{43}$$

In electrotechnology, if the input of a two-port network has the complex amplitude S_1, and its output, S_2, then the logarithm of the ratio S_1/S_2 is called the "transfer exponent" of the network. (For a transmission line, the transfer exponent per unit length is called the "propagation coefficient.") Publication 27–2 of the International Electrotechnical Commission expresses the transfer exponent as $\Gamma = A + jB$, with a footnote on the unit of Γ that, in general, units are used only with A and B separately. Thus A is expressed in nepers, and B in radians, and Γ involves a mixture of units. Equations (38) and (43) show that Γ can be considered to be dimensionally homogeneous, being expressed as a complex number times a radian or a neper.

Formulas (42) for $f(\theta)$ and $f(L)$ involve the explicit appearance of units of θ and l, a feature that is undesirable in quantity relations. This situation arises from the original stipulation that $f(X)$ be numerical for class C functions, and is associated with the definition

$$e^x = \exp x \triangleq \sum_0^\infty x^n/n! \tag{44}$$

and the Euler formula

$$e^{jx} = \cos x + j \sin x \tag{45}$$

for *numerical* x.

Equation (45) can be generalized [3] to the *quantity* equation

$$e^{j\theta} = \cos \theta + j \sin \theta \tag{46}$$

if we assign the dimensional properties

$$\begin{aligned} D^2(\theta) &= D(1) \\ D(\sin \theta) &= D(\theta) \\ D(\cos \theta) &= D(1) \end{aligned} \tag{47}$$

The function $e^{j\theta}$ becomes a dimensional hybrid, i.e., it is a vector having components in two dimension spaces. This mixed dimensionality is, however, admissible in (18a), since (47) implies that elements of the form $a + jb\theta$ comprise a multiplicative group.

Analogously,

$$e^L = \cosh L + \sinh L \tag{48}$$

with

$$\begin{aligned} D^2(L) = D^2(\theta) &= D(1) \\ D(\sinh L) &= D(L) \\ D(\cosh L) &= D(1) \end{aligned} \tag{49}$$

In general, the exponential function of a nonnumeric quantity provides an artifice for the simultaneous treatment of differentially related quantities of different dimensions. Those hybrid quantities comprise a hypercomplex algebra on a subspace of the vector space of dimensions; they are elements of a division ring, and isomorphic to 2×2 matrices.

6. Conclusions

Angle and *logarithm* are interrelated dimensionally, as well as analytically. When *radian* is considered a base unit, *neper* becomes a derived unit, and should be considered the SI unit for logarithmic quantities.

7. References

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Complete Elliptic Integrals Resulting from Infinite Integrals of Bessel Functions. II*

Shigehiko Okui**

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Infinite integrals involving products of Bessel and trigonometric or hyperbolic functions reducible to complete elliptic integrals are compiled. The table also contains certain types of infinite double integrals of modified Bessel functions. All the results are expressed in conveniently compact form suited for practical applications.

Key words: Bessel functions; complete elliptic integrals; hyperbolic functions; infinite integrals; modified Bessel functions; noise theory; signal statistics; trigonometric functions.

1. Introduction

This table is primarily intended for applied mathematicians, physicists and engineers faced with the problem of evaluating infinite integrals involving Bessel functions.

In a previous paper,[1] infinite integrals of Bessel functions and their products were compiled in conveniently closed form. In this paper, a table is prepared for those involving products of Bessel and trigonometric or hyperbolic functions. It also contains certain kinds of infinite double integrals of modified Bessel functions, which are often encountered especially in the fields of signal statistics and noise theory.

Throughout this paper, the parameters are usually positive real and notations occurring several times on a section are explained at the top of the section.

2. Integrands Involving Products of Bessel and Trigonometric Functions

2.1. $k_1^2=\dfrac{a+b}{2a}$, $\quad k_1^2+k_1'^2=1$

$$k_2^2=\frac{2a}{a+b}$$

$$\int_0^\infty J_0(ax^2)\cos(bx^2)\,dx=\frac{1}{2\sqrt{\pi a}}[K(k_1)+K(k_1')], \qquad a>b;$$

$$=\frac{k_2}{2\sqrt{\pi a}}K(k_2), \qquad a<b. \tag{1}$$

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**Present address: Kobe City Technical College, Tarumi-ku, Kobe, Japan.

[1] Okui, S., Complete elliptic integrals resulting from infinite integrals of Bessel functions, J. Res. Nat. Bur. Stand. (U.S.), 78B (Math. Sci.), No. 3, 113–135 (July–Sept. 1974).

$$\int_0^\infty J_1(ax^2)\cos(bx^2)\,dx = \frac{1}{2\sqrt{\pi a}}\left[2E(k_1) - K(k_1) + 2E(k_1') - K(k_1')\right], \qquad a > b;$$

$$= -\frac{1}{2k_2\sqrt{\pi a}}\left[(2-k_2^2)K(k_2) - 2E(k_2)\right], \qquad a < b. \tag{2}$$

$$\int_0^\infty J_1(ax^2)\cos(bx^2)\,x^{-2}dx = \frac{2\sqrt{a}}{3\sqrt{\pi}}[(1-k_1^2)K(k_1) - (1-2k_1^2)E(k_1)$$

$$+ (1-k_1'^2)K(k_1') - (1-2k_1'^2)E(k_1')], \qquad a > b;$$

$$= \frac{2\sqrt{a}}{3k_2^3\sqrt{\pi}}[(2-k_2^2)E(k_2) - 2(1-k_2^2)K(k_2)], \qquad a < b. \tag{3}$$

$$\int_0^\infty J_2(ax^2)\cos(bx^2)\,dx = -\frac{1}{6\sqrt{\pi a}}[8(1-2k_1^2)E(k_1) - (5-8k_1^2)K(k_1)$$

$$+ 8(1-2k_1'^2)E(k_1') - (5-8k_1'^2)K(k_1')], \qquad a > b;$$

$$= -\frac{1}{6k_2^3\sqrt{\pi a}}[(4-k_2^2)(4-3k_2^2)K(k_2) - 8(2-k_2^2)E(k_2)], \qquad a < b. \tag{4}$$

$$\int_0^\infty J_2(ax^2)\cos(bx^2)\,x^{-2}dx = \frac{2\sqrt{a}}{15\sqrt{\pi}}[(1-k_1^2)(1-8k_1^2)K(k_1) - (1-16k_1^2+16k_1^4)E(k_1)$$

$$+ (1-k_1'^2)(1-8k_1'^2)K(k_1') - (1-16k_1'^2+16k_1'^4)E(k_1')], \qquad a > b;$$

$$= -\frac{2\sqrt{a}}{15k_2^5\sqrt{\pi}}[(16-16k_2^2+k_2^4)E(k_2) - 8(1-k_2^2)(2-k_2^2)K(k_2)], \qquad a < b. \tag{5}$$

$$\int_0^\infty J_2(ax^2)\cos(bx^2)\,x^{-4}\,dx$$

$$= \frac{8\sqrt{a^3}}{105\sqrt{\pi}}[(1-k_1^2)(2+5k_1^2-8k_1^4)K(k_1) - 2(1-2k_1^2)(1+4k_1^2-4k_1^4)E(k_1)$$

$$+ (1-k_1'^2)(2+5k_1'^2-8k_1'^4)K(k_1') - 2(1-2k_1'^2)$$

$$\times (1+4k_1'^2-4k_1'^4)E(k_1')], \qquad a > b;$$

$$= \frac{8\sqrt{a^3}}{105k_2^7\sqrt{\pi}}[(1-k_2^2)(16-16k_2^2-k_2^4)K(k_2) - 2(2-k_2^2)$$

$$\times (4-4k_2^2-k_2^4)E(k_2)], \qquad a < b. \tag{6}$$

$$\int_0^\infty J_3(ax^2)\cos(bx^2)\,dx$$

$$= -\frac{1}{30\sqrt{\pi a}}[2(23-128k_1^2+128k_1^4)E(k_1) - (31-144k_1^2+128k_1^4)K(k_1)$$

$$+2(23-128k_1'^2+128k_1'^4)E(k_1')-(31-144k_1'^2$$

$$+128k_1'^4)K(k_1')\,],\qquad a>b;$$

$$=\frac{1}{30k_2^5\sqrt{\pi a}}\,[\,(2-k_2^2)\,(128-128k_2^2+15k_2^4)K(k_2)-2(128$$

$$-128k_2^2+23k_2^4)E(k_2)\,],\qquad a<b. \tag{7}$$

$$\int_0^\infty J_3(ax^2)\cos(bx^2)\,x^{-2}dx=-\frac{2\sqrt{a}}{105\sqrt{\pi}}[(1-k_1^2)(3-80k_1^2+128k_1^4)K(k_1)-(3-134k_1^2$$

$$+384k_1^4-256k_1^6)E(k_1)+(1-k_1'^2)(3-80k_1'^2+128k_1'^4)K(k_1')$$

$$-(3-134k_1'^2+384k_1'^4-256k_1'^6)E(k_1')\,],\quad a>b;$$

$$=-\frac{2\sqrt{a}}{105k_2^7\sqrt{\pi}}[(2-k_2^2)(128-128k_2^2+3k_2^4)E(k_2)-2(1-k_2^2)$$

$$\times(128-128k_2^2+27k_2^4)K(k_2)],\qquad a<b. \tag{8}$$

$$\int_0^\infty J_3(ax^2)\cos(bx^2)\,x^{-4}dx=\frac{8\sqrt{a^3}}{945\sqrt{\pi}}[(1-k_1^2)(2+15k_1^2-144k_1^4+128k_1^6)K(k_1)-2(1+7k_1^2$$

$$-135k_1^4+256k_1^6-128k_1^8)E(k_1)+(1-k_1'^2)(2+15k_1'^2-144k_1'^4$$

$$+128k_1'^6)K(k_1')-2(1+7k_1'^2-135k_1'^4+256k_1'^6-128k_1'^8)E(k_1')],$$

$$a>b;$$

$$=-\frac{8\sqrt{a^3}}{945k_2^9\sqrt{\pi}}[(1-k_2^2)(2-k_2^2)(128-128k_2^2-k_2^4)K(k_2)-(256$$

$$-512k_2^2+270k_2^4-14k_2^6-2k_2^8)E(k_2)],\qquad a<b. \tag{9}$$

$$\int_0^\infty J_0(ax^2)\,\sin\,(bx^2)\,dx=\frac{1}{2\sqrt{\pi a}}\,[K(k_1)-K(k_1')\,],\qquad a>b;$$

$$=\frac{k_2}{2\sqrt{\pi a}}K(k_2),\qquad a<b. \tag{10}$$

$$\int_0^\infty J_0(ax^2)\,\sin\,(bx^2)\,x^{-2}\,dx=\frac{2\sqrt{a}}{\sqrt{\pi}}\,[E(k_1)-(1-k_1^2)K(k_1)$$

$$+(1-k_1'^2)K(k_1')-E(k_1')],\qquad a>b;$$

$$= \frac{2\sqrt{a}}{k_2\sqrt{\pi}} E(k_2), \qquad a < b. \tag{11}$$

$$\int_0^\infty J_1(ax^2)\sin(bx^2)\,dx = -\frac{1}{2\sqrt{\pi a}}[2E(k_1) - K(k_1) + K(k_1') - 2E(k_1')], \qquad a > b;$$

$$= \frac{1}{2k_2\sqrt{\pi a}}[(2-k_2^2)K(k_2) - 2E(k_2)], \qquad a < b. \tag{12}$$

$$\int_0^\infty J_1(ax^2)\sin(bx^2)\,x^{-2}dx = \frac{2\sqrt{a}}{3\sqrt{\pi}}[(1-k_1^2)K(k_1) - (1-2k_1^2)E(k_1)$$

$$+ (1-2k_1'^2)E(k_1') - (1-k_1'^2)K(k_1')], \qquad a > b;$$

$$= \frac{2\sqrt{a}}{3k_2^3\sqrt{\pi}}[(2-k_2^2)E(k_2) - 2(1-k_2^2)K(k_2)], \qquad a < b. \tag{13}$$

$$\int_0^\infty J_1(ax^2)\sin(bx^2)\,x^{-4}dx = \frac{8\sqrt{a^3}}{15\sqrt{\pi}}[2(1-k_1^2+k_1^4)E(k_1) - (1-k_1^2)(2-k_1^2)K(k_1)$$

$$+ (1-k_1'^2)(2-k_1'^2)K(k_1') - 2(1-k_1'^2+k_1'^4)E(k_1')], \qquad a > b;$$

$$= \frac{8\sqrt{a^3}}{15k_2^5\sqrt{\pi}}[2(1-k_2^2+k_2^4)E(k_2) - (1-k_2^2)(2-k_2^2)K(k_2)], \qquad a < b. \tag{14}$$

$$\int_0^\infty J_2(ax^2)\sin(bx^2)\,dx = -\frac{1}{6\sqrt{\pi a}}[8(1-2k_1^2)E(k_1) - (5-8k_1^2)K(k_1)$$

$$+ (5-8k_1'^2)K(k_1') - 8(1-2k_1'^2)E(k_1')], \qquad a > b;$$

$$= -\frac{1}{6k_2^3\sqrt{\pi a}}[(4-k_2^2)(4-3k_2^2)K(k_2) - 8(2-k_2^2)E(k_2)], \qquad a < b. \tag{15}$$

$$\int_0^\infty J_2(ax^2)\sin(bx^2)\,x^{-2}dx = -\frac{2\sqrt{a}}{15\sqrt{\pi}}[(1-k_1^2)(1-8k_1^2)K(k_1) - (1-16k_1^2+16k_1^4)E(k_1)$$

$$+ (1-16k_1'^2+16k_1'^4)E(k_1')$$

$$- (1-k_1'^2)(1-8k_1'^2)K(k_1')], \qquad a > b;$$

$$= \frac{2\sqrt{a}}{15k_2^5\sqrt{\pi}}[(16-16k_2^2+k_2^4)E(k_2)$$

$$-8(1-k_2^2)(2-k_2^2)K(k_2)], \qquad a < b. \tag{16}$$

$$\int_0^\infty J_2(ax^2)\sin(bx^2)\,x^{-4}dx = \frac{8\sqrt{a^3}}{105\sqrt{\pi}}[(1-k_1^2)(2+5k_1^2-8k_1^4)K(k_1)$$

$$-2(1-2k_1^2)(1+4k_1^2-4k_1^4)E(k_1) +$$

$$+2(1-2k_1'^2)(1+4k_1'^2-4k_1'^4)E(k_1')-(1-k_1'^2)$$
$$\times(2+5k_1'^2-8k_1'^4)K(k_1')], \qquad a>b;$$
$$=\frac{8\sqrt{a^3}}{105k_2^7\sqrt{\pi}}[(1-k_2^2)(16-16k_2^2-k_2^4)K(k_2)-2(2-k_2^2)$$
$$\times(4-4k_2^2-k_2^4)E(k_2)], \qquad a<b. \tag{17}$$

$$\int_0^\infty J_3(ax^2)\sin(bx^2)dx=\frac{1}{30\sqrt{\pi a}}[2(23-128k_1^2+128k_1^4)E(k_1)$$
$$-(31-144k_1^2+128k_1^4)K(k_1)$$
$$+(31-144k_1'^2+128k_1'^4)K(k_1')-2(23-128k_1'^2$$
$$+128k_1'^4)E(k_1')], \qquad a>b;$$

$$=-\frac{1}{30k_2^3\sqrt{\pi a}}[(2-k_2^2)(128-128k_2^2+15k_2^4)K(k_2)$$
$$-2(128-128k_2^2+23k_2^4)E(k_2)], \qquad a<b. \tag{18}$$

$$\int_0^\infty J_3(ax^2)\sin(bx^2)x^{-2}dx=-\frac{2\sqrt{a}}{105\sqrt{\pi}}[(1-k_1^2)(3-80k_1^2+128k_1^4)K(k_1)$$
$$-(3-134k_1^2+384k_1^4-256k_1^6)E(k_1)$$
$$+(3-134k_1'^2+384k_1'^4-256k_1'^6)E(k_1')$$
$$-(1-k_1'^2)(3-80k_1'^2+128k_1'^4)K(k_1')], \qquad a>b;$$

$$=-\frac{2\sqrt{a}}{105k_2^7\sqrt{\pi}}[(2-k_2^2)(128-128k_2^2+3k_2^4)E(k_2)$$
$$-2(1-k_2^2)(128-128k_2^2+27k_2^4)K(k_2)], \quad a<b. \tag{19}$$

$$\int_0^\infty J_3(ax^2)\sin(bx^2)x^{-4}dx=-\frac{8\sqrt{a^3}}{945\sqrt{\pi}}[(1-k_1^2)(2+15k_1^2-144k_1^4+128k_1^6)K(k_1)$$
$$-2(1+7k_1^2-135k_1^4+256k_1^6-128k_1^8)E(k_1)$$
$$+2(1+7k_1'^2-135k_1'^4+256k_1'^6-128k_1'^8)E(k_1')$$
$$-(1-k_1'^2)(2+15k_1'^2-144k_1'^4+128k_1'^6)K(k_1')], \qquad a>b;$$

$$=\frac{8\sqrt{a^3}}{945k_2^9\sqrt{\pi}}[(1-k_2^2)(2-k_2^2)(128-128k_2^2-k_2^4)K(k_2)$$
$$-(256-512k_2^2+270k_2^4-14k_2^6-2k_2^8)E(k_2)], \qquad a<b. \tag{20}$$

2.2.[2] $k^2=\dfrac{2(\sqrt{b+a}-\sqrt{b-a})}{2\sqrt{b}+\sqrt{b+a}-\sqrt{b-a}}, \qquad a<b$

$$\int_0^\infty J_0^2(ax^2)\cos(2bx^2)dx=\frac{k^2}{\sqrt{\pi^3}(\sqrt{b+a}-\sqrt{b-a})}[K(k)]^2. \tag{1}$$

[2] The results are also valid for the integrals involving $\sin(2bx^2)$ in place of $\cos(2bx^2)$.

$$\int_0^\infty J_1^2(ax^2)\cos(2bx^2)dx = -\frac{a^2k^2}{3\sqrt{\pi^3}(1-k^2)^2(\sqrt{b+a}-\sqrt{b-a})^5} \times [(2-k^2)E(k)-2(1-k^2)K(k)]^2. \quad (2)$$

$$\int_0^\infty J_2^2(ax^2)\cos(2bx^2)dx = \frac{a^4k^2}{105\sqrt{\pi^3}(1-k^2)^4(\sqrt{b+a}-\sqrt{b-a})^9} \times [(1-k^2)(16-16k^2-k^4)K(k)-2(2-k^2)(4-4k^2-k^4)E(k)]^2. \quad (3)$$

2.3. $k^2 = \frac{4ab}{(a+b)^2}$

$$\int_0^\infty J_1^2(ax)\sin(2bx)x^{-2}dx = b - \frac{2(a+b)}{3\pi a^2}[(a^2+b^2)E(k)-(a-b)^2K(k)]. \quad (1)$$

$$\int_0^\infty J_2^2(ax)\sin(2bx)x^{-2}dx = \frac{b}{2} - \frac{2(a+b)}{15\pi a^4}[(a-b)^2(4b^2-a^2)K(k)+(a^4+9a^2b^2-4b^4)E(k)]. \quad (2)$$

$$\int_0^\infty J_2(ax)J_1(ax)\sin(2bx)x^{-1}dx = \frac{b}{a} + \frac{1}{3\pi a^3}[(b-a)(4b^2-a^2)K(k)-(a+b)(a^2+4b^2)E(k)]. \quad (3)$$

$$\int_0^\infty J_3(ax)J_1(ax)\sin(2bx)dx = \frac{4b}{a^2} - \frac{1}{3\pi a^4(a+b)}[(4a^4-17a^2b^2+16b^4)K(k)-(7a^2+16b^2)(a+b)^2E(k)]. \quad (4)$$

$$\int_0^\infty J_3(ax)J_2(ax)\sin(2bx)x^{-1}dx = \frac{b}{a} + \frac{1}{15\pi a^5}[(b-a)(3a^4-20a^2b^2+32b^4)K(k)+(a+b)(3a^4+52a^2b^2-32b^4)E(k)]. \quad (5)$$

2.4.[3] $k_1^2 = \frac{4ab}{(a+b)^2-c^2}$, $k_2^2 = \frac{(a+b)^2-c^2}{4ab}$

$a > b$

$$\int_0^\infty J_0(ax)J_0(bx)\cos(cx)dx = \frac{k_1}{\pi\sqrt{ab}}K(k_1), \qquad 0 < c < a-b;$$

$$= \frac{1}{\pi\sqrt{ab}}K(k_2), \qquad a-b < c < a+b.$$

OF 64 (1)

[3] For $c > a+b$ the integrals vanish.

$$\int_0^\infty J_1(ax)J_1(bx)\cos(cx)dx=\frac{1}{\pi k_1\sqrt{ab}}[(2-k_1^2)K(k_1)-2E(k_1)],\qquad 0<c<a-b;$$

$$=\frac{1}{\pi\sqrt{ab}}[K(k_2)-2E(k_2)],\qquad a-b<c<a+b. \tag{2}$$

$$\int_0^\infty J_2(ax)J_2(bx)\cos(cx)dx=\frac{1}{3\pi k_1^3\sqrt{ab}}[(16-16k_1^2+3k_1^4)K(k_1)-8(2-k_1^2)E(k_1)],$$

$$0<c<a-b;$$

$$=\frac{1}{3\pi\sqrt{ab}}[8(1-2k_2^2)E(k_2)-(5-8k_2^2)K(k_2)],$$

$$a-b<c<a+b. \tag{3}$$

$$\int_0^\infty J_3(ax)J_3(bx)\cos(cx)dx=\frac{1}{15\pi k_1^5\sqrt{ab}}[(128-128k_1^2+15k_1^4)(2-k_1^2)K(k_1)$$

$$-2(128-128k_1^2+23k_1^4)E(k_1)],\qquad 0<c<a-b;$$

$$=\frac{1}{15\pi\sqrt{ab}}[(31-144k_2^2+128k_2^4)K(k_2)$$

$$-2(23-128k_2^2+128k_2^4)E(k_2)],\qquad a-b<c<a+b. \tag{4}$$

2.5.[4] $\quad k_1^2=\dfrac{c^2-(a-b)^2}{4ab},\qquad k_2^2=\dfrac{4ab}{c^2-(a-b)^2}$

$$a>b$$

$$\int_0^\infty J_0(ax)J_0(bx)\sin(cx)dx=\frac{1}{\pi\sqrt{ab}}K(k_1),\qquad a-b<c<a+b;$$

$$=\frac{k_2}{\pi\sqrt{ab}}K(k_2),\qquad c>a+b. \qquad \text{OF 169} \tag{1}$$

$$\int_0^\infty J_1(ax)J_1(bx)\sin(cx)dx=-\frac{1}{\pi\sqrt{ab}}[K(k_1)-2E(k_1)],\qquad a-b<c<a+b;$$

$$=-\frac{1}{\pi k_2\sqrt{ab}}[(2-k_2^2)K(k_2)-2E(k_2)],\qquad c>a+b. \tag{2}$$

$$\int_0^\infty J_2(ax)J_2(bx)\sin(cx)dx=\frac{1}{3\pi\sqrt{ab}}[8(1-2k_1^2)E(k_1)$$

$$-(5-8k_1^2)K(k_1)],\qquad a-b<c<a+b;$$

$$=\frac{1}{3\pi k_2^3\sqrt{ab}}[(16-16k_2^2+3k_2^4)K(k_2)-$$

[4] For $0<c<a-b$ the integrals vanish.

$$-8(2-k_2^2)E(k_2)], \qquad c > a+b. \tag{3}$$

$$\int_0^\infty J_3(ax)J_3(bx)\sin(cx)dx = -\frac{1}{15\pi\sqrt{ab}}[(31-144k_1^2+128k_1^4)K(k_1)$$

$$-2(23-128k_1^2+128k_1^4)E(k_1)], \qquad a-b<c<a+b;$$

$$= -\frac{1}{15\pi k_2^5\sqrt{ab}}[(128-128k_2^2+15k_2^4)(2-k_2^2)K(k_2)$$

$$-2(128-128k_2^2+23k_2^4)E(k_2)], \qquad c > a+b. \tag{4}$$

2.6. $k_1^2 = \dfrac{4ab}{(a+b)^2-c^2}$, $\quad k_2^2 = \dfrac{(a+b)^2-c^2}{4ab}$

$$k_3^2 = \frac{c^2-(a+b)^2}{c^2-(a-b)^2}, \qquad a > b$$

$$\int_0^\infty Y_0(ax)Y_0(bx)\cos(cx)dx = \frac{k_1}{\pi\sqrt{ab}}K(k_1), \qquad 0 < c < a-b;$$

$$= \frac{1}{\pi\sqrt{ab}}K(k_2), \qquad a-b < c < a+b;$$

$$= \frac{2\sqrt{1-k_3^2}}{\pi\sqrt{ab}}K(k_3), \qquad c > a+b. \qquad \text{OF 70} \tag{1}$$

2.7.[5] $k_1^2 = \dfrac{c^2-(a-b)^2}{4ab}$, $\quad k_2^2 = \dfrac{4ab}{c^2-(a-b)^2}$

$$a > b$$

$$\int_0^\infty Y_0(ax)J_0(bx)\cos(cx)dx = -\frac{1}{\pi\sqrt{ab}}K(k_1), \qquad a-b < c < a+b;$$

$$= -\frac{k_2}{\pi\sqrt{ab}}K(k_2), \qquad c > a+b. \qquad \text{OF 68} \tag{1}$$

$$\int_0^\infty Y_1(ax)J_1(bx)\cos(cx)dx = \frac{1}{\pi\sqrt{ab}}[K(k_1)-2E(k_1)], \qquad a-b < c < a+b;$$

$$= \frac{1}{\pi k_2\sqrt{ab}}[(2-k_2^2)K(k_2)-2E(k_2)], \qquad c > a+b. \tag{2}$$

$$\int_0^\infty Y_2(ax)J_2(bx)\cos(cx)dx = -\frac{1}{3\pi\sqrt{ab}}[8(1-2k_1^2)E(k_1)-(5-8k_1^2)K(k_1)],$$

$$a-b < c < a+b;$$

---

[5] For $0 < c < a-b$ the integrals vanish.

$$= -\frac{1}{3\pi k_2^3\sqrt{ab}}\left[(16-16k_2^2+3k_2^4)\,K(k_2)-8(2-k_2^2)E(k_2)\right],$$

$$c > a+b. \qquad (3)$$

$$\int_0^\infty Y_3(ax)J_3(bx)\cos(cx)\,dx = \frac{1}{15\pi\sqrt{ab}}\left[(31-144k_1^2+128k_1^4)K(k_1)\right.$$

$$\left.-2(23-128k_1^2+128k_1^4)E(k_1)\right], \quad a-b<c<a+b;$$

$$= \frac{1}{15\pi k_2^5\sqrt{ab}}\left[(2-k_2^2)(128-128k_2^2+15k_2^4)K(k_2)\right.$$

$$\left.-2(128-128k_2^2+23k_2^4)E(k_2)\right], \quad c>a+b. \qquad (4)$$

2.8. $k_1^2=\dfrac{(a-b)^2-c^2}{(a+b)^2-c^2}$. $\quad k_2^2=\dfrac{c^2-(a-b)^2}{4ab}$

$$k_3^2=\frac{4ab}{c^2-(a-b)^2}, \quad a>b$$

$$\int_0^\infty J_0(ax)Y_0(bx)\cos(cx)\,dx=-\frac{2\sqrt{1-k_1^2}}{\pi\sqrt{ab}}K(k_1), \quad 0<c<a-b;$$

$$=-\frac{1}{\pi\sqrt{ab}}K(k_2), \quad a-b<c<a+b;$$

$$=-\frac{k_3}{\pi\sqrt{ab}}K(k_3), \quad c>a+b. \qquad \text{OF 68} \qquad (1)$$

$$\int_0^\infty J_1(ax)Y_1(bx)\cos(cx)\,dx=-\frac{2}{\pi\sqrt{ab(1-k_1^2)}}\left[2E(k_1)-(1-k_1^2)K(k_1)\right], \quad 0<c<a-b;$$

$$=\frac{1}{\pi\sqrt{ab}}\left[K(k_2)-2E(k_2)\right], \quad a-b<c<a+b;$$

$$=\frac{1}{\pi k_3\sqrt{ab}}\left[(2-k_3^2)K(k_3)-2E(k_3)\right], \quad c>a+b. \qquad (2)$$

$$\int_0^\infty J_2(ax)Y_2(bx)\cos(cx)\,dx=-\frac{2}{3\pi\sqrt{ab(1-k_1^2)^3}}\left[8(1+k_1^2)E(k_1)\right.$$

$$\left.-(1-k_1^2)(5+3k_1^2)K(k_1)\right], \quad 0<c<a-b;$$

$$=-\frac{1}{3\pi\sqrt{ab}}\left[8(1-2k_2^2)E(k_2)-(5-8k_2^2)K(k_2)\right], \; a-b<c<a+b;$$

$$=-\frac{1}{3\pi k_3^3\sqrt{ab}}\left[(16-16k_3^2+3k_3^4)K(k_3)\right.$$

$$\left.-8(2-k_3^2)E(k_3)\right], \quad c>a+b. \qquad (3)$$

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$$\int_0^\infty J_3(ax)Y_3(bx)\cos(cx)dx=-\frac{2}{15\pi\sqrt{ab(1-k_1^2)^5}}[2(23+82k_1^2+23k_1^4)E(k_1)$$

$$-(1-k_1^2)(31+82k_1^2+15k_1^4)K(k_1)], \qquad 0<c<a-b;$$

$$=\frac{1}{15\pi\sqrt{ab}}[(31-144k_2^2+128k_2^4)K(k_2)$$

$$-2(23-128k_2^2+128k_2^4)E(k_2)], \qquad a-b<c<a+b;$$

$$=\frac{1}{15\pi k_3^5\sqrt{ab}}[(128-128k_3^2+15k_3^4)(2-k_3^2)K(k_3)$$

$$-2(128-128k_3^2+23k_3^4)E(k_3)], \qquad c>a+b. \tag{4}$$

2.9. $k^2=1-\frac{a^2}{b^2}$

$$\int_0^\infty Y_0(ax)J_0(ax)\sin(2bx)dx=-\frac{1}{\pi b}K(k), \qquad a<b;$$

$$=0, \qquad a>b. \qquad \text{OF 172} \quad (1)$$

2.10. $k^2=\frac{b^2}{a^2}$, $\quad a>b$

$$\int_0^\infty [Y_0(ax)J_1(ax)+J_0(ax)Y_1(ax)]\sin(2bx)dx=-\frac{2b}{\pi a^2}K(k). \tag{1}$$

2.11. $k^2=\frac{c^2-(a+b)^2}{c^2-(a-b)^2}$

$$\int_0^\infty [Y_0(ax)J_0(bx)+J_0(ax)Y_0(bx)]\sin(cx)dx=-\frac{2\sqrt{1-k^2}}{\pi\sqrt{ab}}K(k), \qquad c>a+b;$$

$$=0, \qquad 0<c<a+b. \qquad \text{OF 172} \quad (1)$$

2.12. $k^2=\frac{\sqrt{a^2+b^2}+b}{2\sqrt{a^2+b^2}}$, $\quad k^2+k'^2=1$

$$\int_0^\infty K_0(ax^2)\cos(bx^2)dx=\frac{\sqrt{\pi kk'}}{2\sqrt{a}}[K(k)+K(k')]. \tag{1}$$

$$\int_0^\infty K_0(ax^2)\cos(bx^2)x^2dx=\frac{\sqrt{\pi(kk')^3}}{2\sqrt{a^3}}[2E(k)-K(k)+2E(k')-K(k')]. \tag{2}$$

$$\int_0^\infty K_0(ax^2)\cos(bx^2)x^4dx=-\frac{\sqrt{\pi(kk')^5}}{2\sqrt{a^5}}[8(1-2k^2)E(k)-(5-8k^2)K(k)$$
$$+8(1-2k'^2)E(k')-(5-8k'^2)K(k')]. \quad (3)$$

$$\int_0^\infty K_0(ax^2)\cos(bx^2)x^6dx=-\frac{\sqrt{\pi(kk')^7}}{2\sqrt{a^7}}[2(23-128k^2+128k^4)E(k)$$
$$-(31-144k^2+128k^4)K(k)+2(23-128k'^2$$
$$+128k'^4)E(k')-(31-144k'^2+128k'^4)K(k')]. \quad (4)$$

$$\int_0^\infty K_1(ax^2)\cos(bx^2)x^2dx=\frac{\sqrt{\pi kk'}}{2\sqrt{a^3}}[(1-k^2)K(k)-(1-2k^2)E(k)$$
$$+(1-k'^2)K(k')-(1-2k'^2)E(k')]. \quad (5)$$

$$\int_0^\infty K_1(ax^2)\cos(bx^2)x^4dx=\frac{\sqrt{\pi(kk')^3}}{2\sqrt{a^5}}[(1-k^2)(1-8k^2)K(k)-(1-16k^2+16k^4)E(k)$$
$$+(1-k'^2)(1-8k'^2)K(k')-(1-16k'^2+16k'^4)E(k')]. \quad (6)$$

$$\int_0^\infty K_1(ax^2)\cos(bx^2)x^6dx=-\frac{\sqrt{\pi(kk')^5}}{2\sqrt{a^7}}[(1-k^2)(3-80k^2+128k^4)K(k)$$
$$-(3-134k^2+384k^4-256k^6)E(k)+(1-k'^2)(3-80k'^2+128k'^4)K(k')$$
$$-(3-134k'^2+384k'^4-256k'^6)E(k')]. \quad (7)$$

$$\int_0^\infty K_2(ax^2)\cos(bx^2)x^4dx=\frac{\sqrt{\pi kk'}}{2\sqrt{a^5}}[(1-k^2)(2+5k^2-8k^4)K(k)$$
$$-2(1-2k^2)(1+4k^2-4k^4)E(k)+(1-k'^2)(2+5k'^2-8k'^4)K(k')$$
$$-2(1-2k'^2)(1+4k'^2-4k'^4)E(k')]. \quad (8)$$

$$\int_0^\infty K_2(ax^2)\cos(bx^2)x^6dx=\frac{\sqrt{\pi(kk')^3}}{2\sqrt{a^7}}[(1-k^2)(2+15k^2-144k^4$$
$$+128k^6)K(k)-2(1+7k^2-135k^4+256k^6-128k^8)E(k)$$
$$+(1-k'^2)(2+15k'^2-144k'^4+128k'^6)K(k')-2(1+7k'^2$$
$$-135k'^4+256k'^6-128k'^8)E(k')]. \quad (9)$$

$$\int_0^\infty K_0(ax^2)\sin(bx^2)dx=\frac{\sqrt{\pi kk'}}{2\sqrt{a}}[K(k)-K(k')]. \quad (10)$$

$$\int_0^\infty K_0(ax^2)\sin(bx^2)x^2dx=-\frac{\sqrt{\pi(kk')^3}}{2\sqrt{a^3}}[2E(k)-K(k)+K(k')-2E(k')]. \quad (11)$$

$$\int_0^\infty K_0(ax^2)\sin(bx^2)x^4dx=-\frac{\sqrt{\pi(kk')^5}}{2\sqrt{a^5}}[8(1-2k^2)E(k)-(5-8k^2)K(k)$$
$$+(5-8k'^2)K(k')-8(1-2k'^2)E(k')]. \quad (12)$$

$$\int_0^\infty K_0(ax^2)\sin(bx^2)x^6dx=\frac{\sqrt{\pi(kk')^7}}{2\sqrt{a^7}}\,[2(23-128k^2+128k^4)\,E(k)$$
$$-(31-144k^2+128k^4)K(k)+(31-144k'^2+128k'^4)K(k')$$
$$-2(23-128k'^2+128k'^4)E(k')]\,. \tag{13}$$

$$\int_0^\infty K_0(ax^2)\sin(bx^2)x^{-2}dx=-\frac{\sqrt{\pi a}}{2\sqrt{kk'}}\,[2E(k)-K(k)+K(k')-2E(k')]\,. \tag{14}$$

$$\int_0^\infty K_1(ax^2)\sin(bx^2)dx=\frac{\sqrt{\pi}}{2\sqrt{akk'}}\,[E(k)-(1-k^2)K(k)+(1-k'^2)K(k')-E(k')]\,. \tag{15}$$

$$\int_0^\infty K_1(ax^2)\sin(bx^2)x^2dx=\frac{\sqrt{\pi kk'}}{2\sqrt{a^3}}\,[(1-k^2)K(k)-(1-2k^2)E(k)$$
$$+(1-2k'^2)E(k')-(1-k'^2)K(k')]\,. \tag{16}$$

$$\int_0^\infty K_1(ax^2)\sin(bx^2)x^4dx=-\frac{\sqrt{\pi(kk')^3}}{2\sqrt{a^5}}\,[(1-k^2)(1-8k^2)K(k)-(1-16k^2+16k^4)E(k)$$
$$+(1-16k'^2+16k'^4)E(k')-(1-k'^2)(1-8k'^2)K(k')]\,. \tag{17}$$

$$\int_0^\infty K_1(ax^2)\sin(bx^2)x^6dx=-\frac{\sqrt{\pi(kk')^5}}{2\sqrt{a^7}}\,[(1-k^2)(3-80k^2+128k^4)K(k)$$
$$-(3-134k^2+384k^4-256k^6)E(k)+(3-134k'^2+384k'^4-256k'^6)E(k')$$
$$-(1-k'^2)(3-80k'^2+128k'^4)K(k')]\,. \tag{18}$$

$$\int_0^\infty K_2(ax^2)\sin(bx^2)x^2dx=\frac{\sqrt{\pi}}{2\sqrt{a^3kk'}}\,[2(1-k^2+k^4)E(k)-(1-k^2)(2-k^2)K(k)$$
$$+(1-k'^2)(2-k'^2)K(k')-2(1-k'^2+k'^4)E(k')]\,. \tag{19}$$

$$\int_0^\infty K_2(ax^2)\sin(bx^2)x^4dx=\frac{\sqrt{\pi kk'}}{2\sqrt{a^5}}\,[(1-k^2)(2+5k^2-8k^4)K(k)$$
$$-2(1-2k^2)(1+4k^2-4k^4)E(k)+2(1-2k'^2)(1+4k'^2-4k'^4)E(k')$$
$$-(1-k'^2)(2+5k'^2-8k'^4)K(k')]\,. \tag{20}$$

$$\int_0^\infty K_2(ax^2)\sin(bx^2)x^6dx=\frac{\sqrt{\pi(kk')^3}}{2\sqrt{a^7}}\,[2(1+7k^2-135k^4+256k^6-128k^8)E(k)$$
$$-(1-k^2)(2+15k^2-144k^4+128k^6)K(k)+(1-k'^2)(2+15k'^2-144k'^4+128k'^6)K(k')$$
$$-2(1+7k'^2-135k'^4+256k'^6-128k'^8)E(k')]\,. \tag{21}$$

2.13. $k^2=1/2\left(1-\dfrac{\sqrt{2b}}{\sqrt{b+\sqrt{a^2+b^2}}}\right)$, $\quad k^2+k'^2=1$

$$\int_0^\infty K_0^2(ax^2)\cos(2bx^2)dx=\frac{\sqrt{\pi}(1-2k^2)}{4\sqrt{b}}\,[K(k')+K(k)]^2\,. \tag{1}$$

$$\int_0^\infty K_1(ax^2)K_0(ax^2)\cos(2bx^2)x^2dx=\frac{(1-2k^2)\sqrt{\pi}}{16kk'\sqrt{b(a^2+b^2)}}[K(k')+K(k)][k^2K(k')$$
$$-(1-2k'^2)E(k')+k'^2K(k)-(1-2k^2)E(k)]. \quad (2)$$

$$\int_0^\infty K_0^2(ax^2)\sin(2bx^2)dx=\frac{\sqrt{\pi}(1-2k^2)}{4\sqrt{b}}[K(k')-K(k)]^2. \quad (3)$$

$$\int_0^\infty K_1(ax^2)K_0(ax^2)\sin(2bx^2)x^2dx=\frac{(1-2k^2)\sqrt{\pi}}{16kk'\sqrt{b(a^2+b^2)}}[K(k')-K(k)][k^2K(k')$$
$$-(1-2k'^2)E(k')-k'^2K(k)+(1-2k^2)E(k)]. \quad (4)$$

2.14. $k^2=1/2\left(1-\frac{\sqrt{2b}}{\sqrt{b+\sqrt{a^2+b^2}}}\right)$, $\quad k^2+k'^2=1$

$$\int_0^\infty K_0(ax^2)I_0(ax^2)\cos(2bx^2)dx=\frac{1-2k^2}{2\sqrt{\pi b}}K(k)[K(k')+K(k)]. \quad (1)$$

$$\int_0^\infty K_0(ax^2)I_0(ax^2)\sin(2bx^2)dx=\frac{1-2k^2}{2\sqrt{\pi b}}K(k)[K(k')-K(k)]. \quad (2)$$

2.15. $k^2=\frac{b^2}{a^2+b^2}$

$$\int_0^\infty K_1(ax)K_0(ax)\cos(2bx)xdx=\frac{\pi k}{4ab}E(k). \quad (1)$$

$$\int_0^\infty K_2(ax)K_1(ax)\cos(2bx)x^3dx=\frac{\pi k}{16ab^3}[(1-k^2)(3-4k^2)K(k)-(3-13k^2+8k^4)E(k)]. \quad (2)$$

$$\int_0^\infty K_1(ax)K_0(ax)\sin(2bx)x^2dx=\frac{\pi k}{8ab^2}[(1-k^2)K(k)-(1-2k^2)E(k)]. \quad (3)$$

$$\int_0^\infty K_2(ax)K_1(ax)\sin(2bx)\,x^4dx=\frac{3\pi k}{32ab^4}$$
$$\times[(1-k^2)(2+5k^2-8k^4)K(k)-2(1-2k^2)(1+4k^2-4k^4)E(k)]. \quad (4)$$

2.16. $k^2=\frac{(a-b)^2+c^2}{(a+b)^2+c^2}$

$$\int_0^\infty K_0(ax)K_0(bx)\cos(cx)dx=\frac{\pi\sqrt{1-k^2}}{2\sqrt{ab}}K(k). \quad \text{OF 88} \quad (1)$$

$$\int_0^\infty K_0(ax)K_0(bx)\sin(cx)xdx=\frac{\pi c\sqrt{(1-k^2)^3}}{8k^2\sqrt{(ab)^3}}[K(k)-E(k)]. \quad (2)$$

$$\int_0^\infty K_1(ax)K_1(bx)\sin(cx)x\,dx=\frac{\pi c\sqrt{1-k^2}}{8k^2\sqrt{(ab)^3}}\left[(1+k^2)E(k)-(1-k^2)K(k)\right]. \tag{3}$$

2.17. $k^2=\dfrac{a^2}{a^2+b^2}$

$$\int_0^\infty K_1(ax)I_2(ax)\cos(2bx)x^{-1}dx=\frac{1}{3k^3}\left[4(1-k^2)K(k)-(4-5k^2)E(k)\right]-\frac{\pi b}{2a}. \tag{1}$$

$$\int_0^\infty K_2(ax)I_3(ax)\cos(2bx)x^{-1}dx=\frac{1}{15k^5}\left[4(1-k^2)(8+k^2)K(k)-(32-12k^2-23k^4)E(k)\right]-\frac{\pi b}{2a}. \tag{2}$$

2.18. $k^2=\dfrac{4ab}{(a+b)^2+c^2}$, $\quad a>b$

$$\int_0^\infty K_0(ax)I_0(bx)\cos(cx)dx=\frac{k}{2\sqrt{ab}}K(k). \qquad \text{OF 87} \tag{1}$$

$$\int_0^\infty K_1(ax)I_0(bx)\cos(cx)xdx=\frac{k}{16(1-k^2)a\sqrt{(ab)^3}}$$
$$\times\left[k^2(a^2-b^2-c^2)E(k)+4ab(1-k^2)K(k)\right]. \tag{2}$$

$$\int_0^\infty K_1(ax)I_1(bx)\cos(cx)dx=\frac{1}{2k\sqrt{ab}}\left[(2-k^2)K(k)-2E(k)\right]. \tag{3}$$

$$\int_0^\infty K_2(ax)I_2(bx)\cos(cx)dx=\frac{1}{6k^3\sqrt{ab}}\left[(4-k^2)(4-3k^2)K(k)-8(2-k^2)E(k)\right]. \tag{4}$$

$$\int_0^\infty K_3(ax)I_3(bx)\cos(cx)dx=\frac{1}{30k^5\sqrt{ab}}$$
$$\times\left[(128-128k^2+15k^4)(2-k^2)K(k)-2(128-128k^2+23k^4)E(k)\right]. \tag{5}$$

$$\int_0^\infty K_4(ax)I_4(bx)\cos(cx)dx=\frac{1}{210k^7\sqrt{ab}}$$
$$\times\left[(6144-12288k^2+8000k^4-1856k^6+105k^8)K(k)-32(2-k^2)(96-96k^2+11k^4)E(k)\right]. \tag{6}$$

$$\int_0^\infty K_0(ax)I_0(bx)\sin(cx)\,x\,dx=\frac{ck^3}{8(1-k^2)\sqrt{(ab)^3}}E(k). \tag{7}$$

$$\int_0^\infty K_1(ax)I_1(bx)\sin(cx)\,x\,dx=\frac{ck}{8(1-k^2)\sqrt{(ab)^3}}\left[(2-k^2)E(k)-2(1-k^2)K(k)\right]. \tag{8}$$

$$\int_0^\infty K_2(ax)I_2(bx)\sin(cx)\,x\,dx=\frac{c}{8k(1-k^2)\sqrt{(ab)^3}}$$
$$\times\left[(16-16k^2+k^4)E(k)-8(1-k^2)(2-k^2)K(k)\right]. \tag{9}$$

$$\int_0^\infty K_3(ax)I_3(bx)\sin(cx)\,x\,dx=\frac{c}{24k^3(1-k^2)\sqrt{(ab)^3}}$$
$$\times[(2-k^2)(128-128k^2+3k^4)E(k)$$
$$-2(1-k^2)(128-128k^2+27k^4)K(k)]. \quad (10)$$

2.19. $k^2=\dfrac{4ab}{(a+b)^2+c^2}=\sin^2\alpha, \qquad \sin\beta=\dfrac{c}{\sqrt{(a-b)^2+c^2}}$

$a>b$

$\Lambda_0(\alpha,\beta)$ is Heuman's Lambda function

$$\int_0^\infty K_0(ax)I_1(bx)\cos(cx)x^{-1}dx=\frac{\sqrt{ab}}{bk}E(k)+\frac{k(b^2-a^2)}{4\sqrt{ab^3}}K(k)-\frac{\pi c}{4b}\Lambda_0(\alpha,\beta). \quad (1)$$

$$\int_0^\infty K_0(ax)I_1(bx)\sin(cx)dx=-\frac{ck}{4\sqrt{ab^3}}K(k)+\frac{\pi}{4b}\Lambda_0(\alpha,\beta). \quad (2)$$

$$\int_0^\infty K_1(ax)I_2(bx)\sin(cx)dx=\frac{c}{k\sqrt{ab^3}}E(k)-\frac{ck(2a^2+b^2+c^2)}{4b\sqrt{(ab)^3}}K(k)+\frac{\pi a}{4b^2}\Lambda_0(\alpha,\beta)+\frac{\pi}{2a}. \quad (3)$$

2.20. $k^2=1/2\left(1-\dfrac{a^2-b^2+c^2}{\sqrt{(a^2-b^2+c^2)^2+4a^2b^2}}\right), \qquad k^2+k'^2=1$

$$\int_0^\infty K_0(ax)J_0(bx)\cos(cx)\,dx=\frac{\sqrt{kk'}}{\sqrt{ab}}K(k). \quad (1)$$

$$\int_0^\infty K_1(ax)J_1(bx)\cos(cx)\,dx=\frac{1}{\sqrt{abkk'}}[E(k)-(1-k^2)K(k)]. \quad (2)$$

$$\int_0^\infty K_2(ax)J_2(bx)\cos(cx)\,dx=\frac{1}{3\sqrt{ab(kk')^3}}$$
$$\times[(1-k^2)(2-3k^2)K(k)-2(1-2k^2)E(k)]. \quad (3)$$

$$\int_0^\infty K_3(ax)J_3(bx)\cos(cx)\,dx=\frac{1}{15\sqrt{ab(kk')^5}}$$
$$\times[(8-23k^2+23k^4)E(k)-(1-k^2)(8-19k^2+15k^4)K(k)]. \quad (4)$$

$$\int_0^\infty K_0(ax)J_0(bx)\sin(cx)\,x\,dx=\frac{c\sqrt{(kk')^3}}{\sqrt{(ab)^3}}[2E(k)-K(k)]. \quad (5)$$

$$\int_0^\infty K_1(ax)J_1(bx)\sin(cx)xdx=\frac{c\sqrt{kk'}}{\sqrt{(ab)^3}}[(1-k^2)K(k)-(1-2k^2)E(k)]. \quad (6)$$

$$\int_0^\infty K_2(ax)J_2(bx)\sin(cx)x\,dx=\frac{c}{\sqrt{(ab)^3kk'}}\left[2(1-k^2+k^4)E(k)-(1-k^2)(2-k^2)K(k)\right]. \tag{7}$$

$$\int_0^\infty K_3(ax)J_3(bx)\sin(cx)x\,dx=\frac{c}{3\sqrt{(abkk')^3}}\left[(8-15k^2+3k^4)(1-k^2)K(k)\right.$$
$$\left.-(8-19k^2+9k^4-6k^6)E(k)\right]. \tag{8}$$

3. Integrands Involving Products of Bessel and Hyperbolic Functions

3.1. $k^2=\frac{a+b}{2a}$, $k^2+k'^2=1$, $a>b$

$$\int_0^\infty K_0(ax^2)\cosh(bx^2)\,dx=\frac{\sqrt{\pi}}{2\sqrt{2a}}\left[K(k)+K(k')\right]. \tag{1}$$

$$\int_0^\infty K_0(ax^2)\cosh(bx^2)\,x^2dx=\frac{\sqrt{\pi}}{4(a^2-b^2)\sqrt{2a}}$$
$$\times\left[2aE(k)-(a-b)K(k)+2aE(k')-(a+b)K(k')\right]. \tag{2}$$

$$\int_0^\infty K_0(ax^2)\cosh(bx^2)\,x^4dx=\frac{\sqrt{\pi}}{8(a^2-b^2)^2\sqrt{2a}}$$
$$\times\left[(a-b)(a-3b)K(k)+8abE(k)+(a+b)(a+3b)K(k')-8abE(k')\right]. \tag{3}$$

$$\int_0^\infty K_0(ax^2)\cosh(bx^2)\,x^6dx=\frac{\sqrt{\pi}}{16(a^2-b^2)^3\sqrt{2a}}$$
$$\times\left[2a(9a^2+23b^2)E(k)-(a-b)(9a^2-8ab+15b^2)K(k)\right.$$
$$\left.+2a(9a^2+23b^2)E(k')-(a+b)(9a^2+8ab+15b^2)K(k')\right]. \tag{4}$$

$$\int_0^\infty K_1(ax^2)\cosh(bx^2)\,x^2dx=\frac{\sqrt{\pi}}{4(a^2-b^2)\sqrt{2a}}$$
$$\times\left[(a-b)K(k)+2bE(k)+(a+b)K(k')-2bE(k')\right]. \tag{5}$$

$$\int_0^\infty K_1(ax^2)\cosh(bx^2)x^4dx=\frac{\sqrt{\pi}}{8(a^2-b^2)^2\sqrt{2a}}\left[2(3a^2+b^2)E(k)\right.$$
$$\left.-(a-b)(3a-b)K(k)+2(3a^2+b^2)E(k')-(a+b)(3a+b)K(k')\right]. \tag{6}$$

$$\int_0^\infty K_1(ax^2)\cosh(bx^2)x^6dx=\frac{\sqrt{\pi}}{16(a^2-b^2)^3\sqrt{2a}}\left[(5a^2-24ab+3b^2)(a-b)K(k)\right.$$
$$\left.+2b(29a^2+3b^2)E(k)+(5a^2+24ab+3b^2)(a+b)K(k')-2b(29a^2+3b^2)E(k')\right]. \tag{7}$$

$$\int_0^\infty K_2(ax^2)\cosh(bx^2)x^4dx=\frac{\sqrt{\pi}}{8(a^2-b^2)^2\sqrt{2a^3}}\left[(5a^2-3ab-4b^2)(a-b)K(k)\right.$$
$$\left.+8b(2a^2-b^2)E(k)+(a+b)(5a^2+3ab-4b^2)K(k')-8b(2a^2-b^2)E(k')\right]. \tag{8}$$

$$\int_0^\infty K_0(ax^2)\sinh(bx^2)dx=\frac{\sqrt{\pi}}{2\sqrt{2a}}[K(k)-K(k')]. \tag{9}$$

$$\int_0^\infty K_0(ax^2)\sinh(bx^2)x^2dx=\frac{\sqrt{\pi}}{4(a^2-b^2)\sqrt{2a}}[2aE(k)-(a-b)K(k)+(a+b)K(k')-2aE(k')]. \tag{10}$$

$$\int_0^\infty K_0(ax^2)\sinh(bx^2)x^4dx=\frac{\sqrt{\pi}}{8(a^2-b^2)^2\sqrt{2a}}[(a-b)(a-3b)K(k)+8abE(k)+8abE(k')-(a+b)(a+3b)K(k')]. \tag{11}$$

$$\int_0^\infty K_0(ax^2)\sinh(bx^2)x^6dx=\frac{\sqrt{\pi}}{16(a^2-b^2)^3\sqrt{2a}}[2a(9a^2+23b^2)E(k)-(a-b)(9a^2-8ab+15b^2)K(k)+(a+b)(9a^2+8ab+15b^2)K(k')-2a(9a^2+23b^2)E(k')]. \tag{12}$$

$$\int_0^\infty K_1(ax^2)\sinh(bx^2)x^2dx=\frac{\sqrt{\pi}}{4(a^2-b^2)\sqrt{2a}}[(a-b)K(k)+2bE(k)+2bE(k')-(a+b)K(k')]. \tag{13}$$

$$\int_0^\infty K_1(ax^2)\sinh(bx^2)x^4dx=\frac{\sqrt{\pi}}{8(a^2-b^2)^2\sqrt{2a}}[2(3a^2+b^2)E(k)-(a-b)(3a-b)K(k)+(a+b)(3a+b)K(k')-2(3a^2+b^2)E(k')]. \tag{14}$$

$$\int_0^\infty K_1(ax^2)\sinh(bx^2)x^6dx=\frac{\sqrt{\pi}}{16(a^2-b^2)^3\sqrt{2a}}[(5a^2-24ab+3b^2)(a-b)K(k)+2b(29a^2+3b^2)E(k)+2b(29a^2+3b^2)E(k')-(5a^2+24ab+3b^2)(a+b)K(k')]. \tag{15}$$

$$\int_0^\infty K_2(ax^2)\sinh(bx^2)x^4dx=\frac{\sqrt{\pi}}{8(a^2-b^2)^2\sqrt{2a^3}}[(5a^2-3ab-4b^2)(a-b)K(k)+8b(2a^2-b^2)E(k)+8b(2a^2-b^2)E(k')-(a+b)(5a^2+3ab-4b^2)K(k')]. \tag{16}$$

3.2. $k^2=\frac{b^2}{a^2}$, $a>b$

$$\int_0^\infty K_1(ax)K_0(ax)\cosh(2bx)xdx=\frac{\pi}{4a^2(1-k^2)}E(k). \tag{1}$$

$$\int_0^\infty K_2(ax)K_1(ax)\cosh(2bx)x^3dx=\frac{\pi}{16a^4k^2(1-k^2)^3}[(3+7k^2-2k^4)E(k)-(1-k^2)(3+k^2)K(k)]. \tag{2}$$

$$\int_0^\infty K_1(ax)K_0(ax)\sinh(2bx)x^2dx=\frac{\pi b}{8a^4k^2(1-k^2)^2}[(1+k^2)E(k)-(1-k^2)K(k)]. \tag{3}$$

$$\int_0^\infty K_2(ax)K_1(ax)\sinh(2bx)\,x^4dx=\frac{3\pi b}{32a^6k^4(1-k^2)^4}$$
$$\times[(1-k^2)(2-9k^2-k^4)K(k)-2(1+k^2)(1-6k^2+k^4)E(k)]. \quad (4)$$

3.3. $k_1^2=\dfrac{(a-b)^2-c^2}{(a+b)^2-c^2}$, $\quad k_2^2=\dfrac{c^2-(a-b)^2}{4ab}$

$a>b$

$$\int_0^\infty K_0(ax)K_0(bx)\cosh(cx)\,dx=\frac{\pi\sqrt{1-k_1^2}}{2\sqrt{ab}}K(k_1), \quad c<a-b;$$
$$=\frac{\pi}{2\sqrt{ab}}K(k_2), \quad a-b<c<a+b. \quad (1)$$

$$\int_0^\infty K_0(ax)K_0(bx)\sinh(cx)\,x\,dx=\frac{\pi c\sqrt{(1-k_1^2)^3}}{8k_1^2\sqrt{(ab)^3}}[K(k_1)-E(k_1)], \quad c<a-b;$$
$$=\frac{\pi c}{8k_2^2(1-k_2^2)\sqrt{(ab)^3}}[E(k_2)-(1-k_2^2)K(k_2)],$$
$$a-b<c<a+b. \quad (2)$$

$$\int_0^\infty K_1(ax)K_1(bx)\sinh(cx)xdx=\frac{\pi c\sqrt{1-k_1^2}}{8k_1^2\sqrt{(ab)^3}}[(1+k_1^2)E(k_1)-(1-k_1^2)K(k_1)], \quad c<a-b;$$
$$=\frac{\pi c}{8k_2^2(1-k_2^2)\sqrt{(ab)^3}}[(1-k_2^2)K(k_2)-(1-2k_2^2)E(k_2)],$$
$$a-b<c<a+b. \quad (3)$$

3.4. $k^2=\dfrac{4ab}{(a+b)^2-c^2}$, $\quad a>b+c$

$$\int_0^\infty K_0(ax)I_0(bx)\cosh(cx)dx=\frac{k}{2\sqrt{ab}}K(k). \quad (1)$$

$$\int_0^\infty K_1(ax)I_1(bx)\cosh(cx)dx=\frac{1}{2k\sqrt{ab}}[(2-k^2)K(k)-2E(k)]. \quad (2)$$

$$\int_0^\infty K_2(ax)I_2(bx)\cosh(cx)dx=\frac{1}{6k^3\sqrt{ab}}[(4-k^2)(4-3k^2)K(k)-8(2-k^2)E(k)]. \quad (3)$$

$$\int_0^\infty K_3(ax)I_3(bx)\cosh(cx)dx=\frac{1}{30k^5\sqrt{ab}}[(128-128k^2+15k^4)(2-k^2)K(k)$$
$$-2(128-128k^2+23k^4)E(k)]. \quad (4)$$

$$\int_0^\infty K_4(ax)I_4(bx)\cosh(cx)dx = \frac{1}{210k^7\sqrt{ab}}\left[(6144 - 12288k^2 + 8000k^4 - 1856k^6 + 105k^8)K(k)\right.$$
$$\left. - 32(2-k^2)(96 - 96k^2 + 11k^4)E(k)\right]. \quad (5)$$

$$\int_0^\infty K_0(ax)I_0(bx)\sinh(cx)xdx = \frac{ck^3}{8(1-k^2)\sqrt{(ab)^3}}E(k). \quad (6)$$

$$\int_0^\infty K_1(ax)I_1(bx)\sinh(cx)xdx = \frac{ck}{8(1-k^2)\sqrt{(ab)^3}}\left[(2-k^2)E(k) - 2(1-k^2)K(k)\right]. \quad (7)$$

$$\int_0^\infty K_2(ax)I_2(bx)\sinh(cx)xdx = \frac{c}{8k(1-k^2)\sqrt{(ab)^3}}\left[(16 - 16k^2 + k^4)E(k)\right.$$
$$\left. - 8(1-k^2)(2-k^2)K(k)\right]. \quad (8)$$

$$\int_0^\infty K_3(ax)I_3(bx)\sinh(cx)xdx = \frac{c}{24k^3(1-k^2)\sqrt{(ab)^3}}\left[(2-k^2)(128 - 128k^2 + 3k^4)E(k)\right.$$
$$\left. - 2(1-k^2)(128 - 128k^2 + 27k^4)K(k)\right]. \quad (9)$$

3.5. $k^2 = 1/2\left(1 - \dfrac{a^2-b^2-c^2}{\sqrt{(a^2-b^2-c^2)^2 + 4a^2b^2}}\right)$, $\quad k^2 + k'^2 = 1$

$a > c$

$$\int_0^\infty K_0(ax)J_0(bx)\cosh(cx)dx = \frac{\sqrt{kk'}}{\sqrt{ab}}K(k). \qquad \text{ET II 15 (23)} \quad (1)$$

$$\int_0^\infty K_1(ax)J_1(bx)\cosh(cx)dx = \frac{1}{\sqrt{abkk'}}\left[E(k) - (1-k^2)K(k)\right]. \quad (2)$$

$$\int_0^\infty K_2(ax)J_2(bx)\cosh(cx)dx = \frac{1}{3\sqrt{ab(kk')^3}}$$
$$\times\left[(1-k^2)(2-3k^2)K(k) - 2(1-2k^2)E(k)\right]. \quad (3)$$

$$\int_0^\infty K_3(ax)J_3(bx)\cosh(cx)dx = \frac{1}{15\sqrt{ab(kk')^5}}$$
$$\times\left[(8 - 23k^2 + 23k^4)E(k) - (1-k^2)(8 - 19k^2 + 15k^4)K(k)\right]. \quad (4)$$

$$\int_0^\infty K_0(ax)J_0(bx)\sinh(cx)xdx = \frac{c\sqrt{(kk')^3}}{\sqrt{(ab)^3}}\left[2E(k) - K(k)\right]. \quad (5)$$

$$\int_0^\infty K_1(ax)J_1(bx)\sinh(cx)xdx = \frac{c\sqrt{kk'}}{\sqrt{(ab)^3}}\left[(1-k^2)K(k) - (1-2k^2)E(k)\right]. \quad (6)$$

$$\int_0^\infty K_2(ax)J_2(bx)\sinh(cx)xdx = \frac{c}{\sqrt{(ab)^3kk'}}\left[2(1-k^2+k^4)E(k) - (1-k^2)(2-k^2)K(k)\right]. \quad (7)$$

$$\int_0^\infty K_3(ax)J_3(bx)\ \sinh\ (cx)xdx = \frac{c}{3\sqrt{(abkk')^3}}\ [(8-15k^2+3k^4)\ (1-k^2)K(k)$$

$$-\ (8-19k^2+9k^4-6k^6)E(k)]. \qquad (8)$$

3.6. $k^2 = 1/2\left(1-\dfrac{a^2-b^2-c^2}{\sqrt{(a^2-b^2-c^2)^2+4a^2b^2}}\right),$

$$cn^2u = \frac{2b^2}{\sqrt{(a^2-b^2-c^2)^2+4a^2b^2}-(a^2-b^2-c^2)},$$

$cn(u)$, $sn(u)$, $dn(u)$ are Jacobian elliptic functions

$$a > c$$

$$\int_0^\infty K_1(ax)J_0(bx)\ \sinh\ (cx)dx = \frac{1}{a}\ [uE(k)-K(k)E(u)$$

$$+\frac{sn(u)dn(u)}{cn(u)}\ K(k)]. \qquad \text{ET II 15 (24)} \qquad (1)$$

4. Miscellaneous Single Integrals

4.1. $k^2 = 1/2\left(1-\dfrac{p^2-a^2+b^2}{\sqrt{(p^2-a^2+b^2)^2+4a^2b^2}}\right),$

$$k^2+k'^2=1, \qquad p > a$$

$$\int_0^\infty e^{-px}I_0(ax)J_0(bx)dx = \frac{2\sqrt{kk'}}{\pi\sqrt{ab}}\ K(k). \qquad (1)$$

$$\int_0^\infty e^{-px}I_0(ax)J_0(bx)xdx = \frac{2p\sqrt{(kk')^3}}{\pi\sqrt{(ab)^3}}\ [2E(k)-K(k)]. \qquad (2)$$

$$\int_0^\infty e^{-px}I_1(ax)J_1(bx)dx = \frac{2}{\pi\sqrt{abkk'}}\ [E(k)-(1-k^2)K(k)]. \qquad (3)$$

$$\int_0^\infty e^{-px}I_1(ax)J_1(bx)xdx = \frac{2p\sqrt{kk'}}{\pi\sqrt{(ab)^3}}\ [(1-k^2)K(k)-(1-2k^2)E(k)]. \qquad (4)$$

$$\int_0^\infty e^{-px}I_2(ax)J_2(bx)dx = \frac{2}{3\pi\sqrt{ab(kk')^3}}\ [(1-k^2)(2-3k^2)K(k)-2(1-2k^2)E(k)]. \qquad (5)$$

$$\int_0^\infty e^{-px}I_2(ax)J_2(bx)xdx = \frac{2p}{\pi\sqrt{(ab)^3kk'}}\ [2(1-k^2+k^4)E(k)-(1-k^2)(2-k^2)K(k)]. \qquad (6)$$

$$\int_0^\infty e^{-px}I_3(ax)J_3(bx)\ dx = \frac{2}{15\pi\ \sqrt{ab(kk')^5}}\ [(8-23k^2+23k^4)E(k)-(1-k^2)(8-19k^2+15k^4)K(k)]. \qquad (7)$$

$$\int_0^\infty e^{-px} I_3(ax) J_3(bx)\, x\, dx = \frac{2p}{3\pi\sqrt{(abkk')^3}} [(8-15k^2+3k^4)(1-k^2)K(k) - (8-19k^2+9k^4-6k^6)E(k)]. \quad (8)$$

4.2. $k^2 = \frac{a^2}{a^2+b^2}$

$$\int_0^\infty [I_0(ax) - \mathbf{L}_0(ax)] J_0(bx)\, dx = \frac{2k}{\pi a} K(k). \quad \text{OB 26 (2.141)} \quad (1)$$

$$\int_0^\infty [I_1(ax) - \mathbf{L}_1(ax)] J_0(bx)\, x^{-1} dx = \frac{2}{\pi k} [K(k) - E(k)]. \quad (2)$$

$$\int_0^\infty [I_2(ax) - \mathbf{L}_2(ax)] J_0(bx)\, x^{-2} dx = \frac{2a}{9\pi k^3} [(2+k^2)K(k) - 2(1+k^2)E(k)]. \quad (3)$$

$$\int_0^\infty [I_3(ax) - \mathbf{L}_3(ax)] J_0(bx)\, x^{-3} dx = \frac{2a^2}{225\pi k^5} [(8+3k^2+4k^4)K(k) - (8+7k^2+8k^4)E(k)]. \quad (4)$$

4.3. $k^2 = \left(\frac{a-b}{a+b}\right)^2$

$$\int_0^\infty [\mathbf{H}_0(ax) - Y_0(ax)] J_0(bx)\, dx = \frac{4}{\pi(a+b)} K(k). \quad \text{OB 26 (2.140)} \quad (1)$$

4.4. $k^2 = \left(\frac{1-a}{1+a}\right)^2$

$$\int_0^\infty e^{-(1/2a^2-1)x^2} I_0\left(\frac{1}{2} a^2 x^2\right) \operatorname{erfc}(x)\, x^2 dx = \frac{2}{(1+a)\sqrt{\pi}} K(k). \quad (1)$$

$$\int_0^\infty e^{-(1/2a^2-1)x^2} \left[I_0\left(\frac{1}{2} a^2 x^2\right) - I_1\left(\frac{1}{2} a^2 x^2\right)\right] \operatorname{erfc}(x)\, x^2 dx = \frac{1}{a^2(1-a^2)\sqrt{\pi}} [(1+a)E(k) - 2aK(k)]. \quad (2)$$

4.5. $k^2 = \frac{1}{1+a^2}$

$$\int_0^\infty e^{-1/2a^2x^2} K_0\left(\frac{1}{2} a^2 x^2\right) \operatorname{erf}(x) x dx = \frac{1}{a^2\sqrt{1+a^2}} [(1+a^2)E(k) - a^2 K(k)]. \quad (1)$$

$$\int_0^\infty e^{-1/2a^2x^2} K_1\left(\frac{1}{2} a^2 x^2\right) \operatorname{erf}(x) x^3 dx = \frac{1}{6a^4\sqrt{1+a^2}} [2a^2(a^2-2)K(k) - (2a^4-3a^2-8)E(k)]. \quad (2)$$

4.6. $k^2=\dfrac{1}{1+2a}$

$$\int_0^\infty e^{-(1+a)x^2}K_0(ax^2)\mathbf{L}_0(x^2)dx=\sqrt{\pi}kK(k). \tag{1}$$

$$\int_0^\infty e^{-(1+a)x^2}K_0(ax^2)\mathbf{L}_1(x^2)x^2dx=\frac{\sqrt{\pi}}{16a^2k}\,[2E(k)-(1-k^2)K(k)]. \tag{2}$$

$$\int_0^\infty e^{-(1+a)x^2}K_0(ax^2)\mathbf{L}_2(x^2)x^4dx=\frac{3\sqrt{\pi}}{512a^4k^3}\,[8(1+k^2)E(k)-(1-k^2)(5+3k^2)K(k)]. \tag{3}$$

4.7. $k^2=\dfrac{a}{1+a}$

$$\int_0^\infty e^{-(1+2a)x^2}K_0(x^2)L_1(2ax^2)\;x^2dx=\frac{\sqrt{\pi}\,k^3}{8\sqrt{2a^3}}\,[2E(k)-K(k)]. \tag{1}$$

$$\int_0^\infty e^{-(1+2a)x^2}K_0(x^2)L_2(2ax^2)\;x^4dx=\frac{3\sqrt{\pi}\,k^7}{128\sqrt{2a^7}}\,[8(1-a)E(k)-(5-3a)K(k)]. \tag{2}$$

$$\int_0^\infty e^{-(1+2a)x^2}K_0(x^2)L_3(2ax^2)\;x^6dx=\frac{5\sqrt{\pi}\,k^{11}}{1024\sqrt{2a^{11}}}$$
$$\times\,[2(23-82a+23a^2)E(k)-(31-82a+15a^2)K(k)]. \tag{3}$$

$$\int_0^\infty e^{-(1+2a)x^2}K_1(x^2)L_1^{(2)}(2ax^2)\;x^6dx=\frac{3\sqrt{\pi}\,k^7}{128\sqrt{2a^{11}}}$$
$$\times\,[(2+9a-a^2)K(k)-2(1-a)(1+6a+a^2)E(k)]. \tag{4}$$

$$\int_0^\infty e^{-(1+2a)x^2}K_1(x^2)L_2^{(2)}(2ax^2)\;x^8dx=\frac{15\sqrt{\pi}\,k^{11}}{2048\sqrt{2a^{15}}}$$
$$\times\,[(2+21a-108a^2+a^3)K(k)-2(1+11a-108a^2+11a^3+a^4)E(k)]. \tag{5}$$

4.8.[6] $k^2=\dfrac{b-a}{b+a}$, $\quad b>a$

$$\int_0^\infty [J_{-1/2}(ax^2)Y_0(ax^2)+J_0(ax^2)Y_{-1/2}(ax^2)]\,J_0(2bx^2)x^2dx=-\sqrt{\frac{2}{\pi^3ab(a+b)}}\,K(k). \tag{1}$$

$$\int_0^\infty [J_{-1/2}(ax^2)Y_1(ax^2)+J_1(ax^2)Y_{-1/2}(ax^2)]\,J_1(2bx^2)x^2dx=-\frac{\sqrt{2(a+b)}}{\sqrt{\pi^3a^3b}}\,[E(k)-\frac{a}{a+b}\,K(k)]. \tag{2}$$

$$\int_0^\infty [J_{-1/2}(ax^2)Y_2(ax^2)+J_2(ax^2)Y_{-1/2}(ax^2)]\,J_2(2bx^2)x^2dx$$
$$=-\frac{1}{3}\sqrt{\frac{2}{\pi^3a^5b(a+b)}}\,[4b(a+b)E(k)-a(4b+a)K(k)]. \tag{3}$$

[6] For $b<a$ the integrals vanish.

4.9. $k^2=\dfrac{b-\sqrt{b^2-a^2}}{2b}, \quad b>a$

$$\int_0^\infty J_{1/4}^2(ax^2)J_{1/4}^2(bx^2)\,x^2dx=\frac{\Gamma(1/4)}{2\pi^2 b\,\sqrt{a}\,\Gamma(3/4)}K(k). \tag{1}$$

4.10. $k^2=\dfrac{b-\sqrt{b^2-a^2}}{2b}, \quad b>a$

$$\int_0^\infty J_{-1/4}(ax^2)Y_{-1/4}(ax^2)J_{-1/4}(bx^2)Y_{-1/4}(bx^2)\,x^2dx=\frac{\Gamma(1/4)}{2\pi^2 b\,\sqrt{a}\,\Gamma(3/4)}K(k). \tag{1}$$

$$\int_0^\infty J_{1/4}(ax^2)Y_{1/4}(ax^2)J_{1/4}(bx^2)Y_{1/4}(bx^2)\,x^4dx=\frac{\Gamma(3/4)\,\sqrt{a}}{4\pi^2 b^3\Gamma(1/4)k^2(1-k^2)(1-2k^2)}$$
$$\times[(1-k^2)K(k)-(1-2k^2)E(k)]. \tag{2}$$

5. Double Integrals

5.1. $k^2=\dfrac{a^2}{pq}, \quad pq>a^2$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(2axy)dxdy=\frac{1}{2\sqrt{pq}}K(k). \tag{1}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(2axy)y^2dxdy=\frac{1}{4(1-k^2)\sqrt{pq^3}}E(k). \tag{2}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(2axy)y^4dxdy=\frac{1}{8(1-k^2)^2\sqrt{pq^5}}[2(2-k^2)E(k)-(1-k^2)K(k)]. \tag{3}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(2axy)y^6dxdy=\frac{1}{16(1-k^2)^3\sqrt{pq^7}}$$
$$\times[(23-23k^2+8k^4)E(k)-4(2-k^2)(1-k^2)K(k)]. \tag{4}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(2axy)y^8dxdy=\frac{1}{32(1-k^2)^4\sqrt{pq^9}}$$
$$\times[8(2-k^2)\,(11-11k^2+6k^4)E(k)-(1-k^2)(71-71k^2+24k^4)K(k)]. \tag{5}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(2axy)\,x^2y^2dxdy=\frac{1}{8(1-k^2)^2\,\sqrt{p^3q^3}}[2E(k)-(1-k^2)K(k)]. \tag{6}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(2axy)\,x^2y^4dxdy=\frac{1}{16(1-k^2)^3\,\sqrt{p^3q^5}}[(7+k^2)E(k)-4(1-k^2)K(k)]. \tag{7}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(2axy)\ x^2y^6dxdy=\frac{1}{32(1-k^2)^4\sqrt{p^3q^7}}$$
$$[2(19+6k^2-k^4)E(k)-(1-k^2)(23+k^2)K(k)]. \tag{8}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(2axy)\ x^2y^8dxdy=\frac{1}{64(1-k^2)^5\sqrt{p^3q^9}}$$
$$\times[(281+142k^2-47k^4+8k^6)E(k)-4(1-k^2)(44+5k^2-k^4)K(k)]. \tag{9}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(2axy)\ x^4y^4dxdy=\frac{3}{32(1-k^2)^4\sqrt{p^5q^5}}$$
$$\times[8(1+k^2)E(k)-(1-k^2)(5+3k^2)K(k)]. \tag{10}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(2axy)\ x^4y^6dxdy=\frac{3}{64(1-k^2)^5\sqrt{p^5q^7}}$$
$$\times[(43+82k^2+3k^4)E(k)-4(1-k^2)(7+9k^2)K(k)]. \tag{11}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(2axy)x^4y^8dxdy=\frac{3}{128(1-k^2)^6\sqrt{p^5q^9}}$$
$$\times[2(158+449k^2+36k^4-3k^6)E(k)-(1-k^2)(211+426k^2+3k^4)K(k)]. \tag{12}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(2axy)x^6y^6dxdy=\frac{15}{128(1-k^2)^6\sqrt{p^7q^7}}$$
$$\times[2(23+82k^2+23k^4)E(k)-(1-k^2)(31+82k^2+15k^4)K(k)]. \tag{13}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(2axy)x^6y^8dxdy=\frac{15}{256(1-k^2)^7\sqrt{p^7q^9}}$$
$$\times[(337+1773k^2+947k^4+15k^6)E(k)-8(1-k^2)(29+118k^2+45k^4)K(k)]. \tag{14}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1(2axy)xydxdy=\frac{1}{4pqk(1-k^2)}[E(k)-(1-k^2)K(k)]. \tag{15}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1(2axy)xy^3dxdy=\frac{1}{8pq^2k(1-k^2)^2}[(1+k^2)E(k)-(1-k^2)K(k)]. \tag{16}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1(2axy)\ xy^5dxdy=\frac{1}{16pq^3k(1-k^2)^3}$$
$$\times[(3+7k^2-2k^4)E(k)-(1-k^2)(3+k^2)K(k)]. \tag{17}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1(2axy)\ xy^7dxdy=\frac{1}{32pq^4k(1-k^2)^4}$$
$$\times[(15+58k^2-33k^4+8k^6)E(k)-(1-k^2)(15+13k^2-4k^4)K(k)]. \tag{18}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1(2axy)\ x^3y^3dxdy=\frac{1}{16p^2q^2k(1-k^2)^3}$$
$$\times[(1+7k^2)E(k)-(1-k^2)(1+3k^2)K(k)]. \tag{19}$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_1(2axy)\ x^3y^5dxdy = \frac{3}{32p^2q^3k(1-k^2)^4}$$
$$\times [(1+14k^2+k^4)E(k) - (1-k^2)(1+7k^2)K(k)]. \quad (20)$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_1(2axy)\ x^3y^7dxdy = \frac{3}{64p^2q^4k(1-k^2)^5}$$
$$\times [(5+108k^2+17k^4-2k^6)E(k) - (1-k^2)(5+58k^2+k^4)K(k)]. \quad (21)$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_1(2axy)x^5y^5dxdy = \frac{3}{64p^3q^3k(1-k^2)^5}$$
$$\times [(3+82k^2+43k^4)E(k) - (1-k^2)(3+46k^2+15k^4)K(k)]. \quad (22)$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_1(2axy)x^5y^7dxdy = \frac{15}{128p^3q^4k(1-k^2)^6}$$
$$\times [(3+125k^2+125k^4+3k^6)E(k) - (1-k^2)(3+74k^2+51k^4)K(k)]. \quad (23)$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_1(2axy)x^{-1}y^{-1}dxdy = \frac{1}{k}[E(k) - (1-k^2)K(k)]. \quad (24)$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_1(2axy)x^{-1}ydxdy = \frac{1}{2qk}[K(k) - E(k)]. \quad (25)$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_1(2axy)x^{-1}y^3dxdy = \frac{1}{4q^2k(1-k^2)}[(1-k^2)K(k) - (1-2k^2)E(k)]. \quad (26)$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_1(2axy)x^{-1}y^5dxdy = \frac{1}{8q^3k(1-k^2)^2}$$
$$\times [(1-k^2)(3-4k^2)K(k) - (3-13k^2+8k^4)E(k)]. \quad (27)$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_1(2axy)x^{-1}y^7dxdy = \frac{1}{16q^4k(1-k^2)^3}[(1-k^2)(15-43k^2+24k^4)K(k)$$
$$-(15-103k^2+128k^4-48k^6)E(k)]. \quad (28)$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_2(2axy)dxdy = \frac{1}{2k^2\sqrt{pq}}[(2-k^2)K(k) - 2E(k)]. \quad (29)$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_2(2axy)y^2dxdy = \frac{1}{4k^2(1-k^2)\sqrt{pq^3}}[(2-k^2)E(k) - 2(1-k^2)K(k)]. \quad (30)$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_2(2axy)y^4dxdy = \frac{1}{8k^2(1-k^2)^2\sqrt{pq^5}}$$
$$\times [2(1-k^2+k^4)E(k) - (1-k^2)(2-k^2)K(k)]. \quad (31)$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_2(2axy)y^6dxdy = \frac{1}{16k^2(1-k^2)^3\sqrt{pq^7}} +$$

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$$\times[(6-9k^2+19k^4-8k^6)E(k)-2(1-k^2)(3-3k^2+2k^4)K(k)]. \quad (32)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(2axy)x^2y^2dxdy=\frac{1}{8k^2(1-k^2)^2\sqrt{p^3q^3}}$$

$$\times[(1-k^2)(2-3k^2)K(k)-2(1-2k^2)E(k)]. \quad (33)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(2axy)x^2y^4dxdy=\frac{1}{16k^2(1-k^2)^3\sqrt{p^3q^5}}$$

$$\times[2(1-k^2)(1-3k^2)K(k)-(2-7k^2-3k^4)E(k)]. \quad (34)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(2axy)x^2y^6dxdy=\frac{3}{32k^2(1-k^2)^4\sqrt{p^3q^7}}$$

$$\times[(1-k^2)(2-9k^2-k^4)K(k)-2(1+k^2)(1-6k^2+k^4)E(k)]. \quad (35)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(2axy)x^4y^4dxdy=\frac{1}{32k^2(1-k^2)^4\sqrt{p^5q^5}}$$

$$\times[(1-k^2)(2-11k^2-15k^4)K(k)-2(1-6k^2-19k^4)E(k)]. \quad (36)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(2axy)x^4y^6dxdy=\frac{3}{64k^2(1-k^2)^5\sqrt{p^5q^7}}$$

$$\times[2(1-k^2)(1-8k^2-25k^4)K(k)-(2-17k^2-108k^4-5k^6)E(k)]. \quad (37)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(2axy)x^4y^8dxdy=\frac{15}{128k^2(1-k^2)^6\sqrt{p^5q^9}}$$

$$\times[(1-k^2)(2-21k^2-108k^4-k^6)K(k)-2(1-11k^2-108k^4-11k^6+k^8)E(k)]. \quad (38)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(2axy)x^{-2}dxdy=\frac{\sqrt{p}}{3k^2\sqrt{q}}[(2-k^2)E(k)-2(1-k^2)K(k)]. \quad (39)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(2axy)x^{-2}y^2dxdy=\frac{\sqrt{p}}{6k^2\sqrt{q^3}}[(2+k^2)K(k)-2(1+k^2)E(k)]. \quad (40)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(2axy)x^{-2}y^4dxdy=\frac{\sqrt{p}}{12k^2(1-k^2)\sqrt{q^5}}$$

$$\times[2(1-k^2)(1+2k^2)K(k)-(2+3k^2-8k^4)E(k)]. \quad (41)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(2axy)x^{-2}y^6dxdy=\frac{\sqrt{p}}{8k^2(1-k^2)^2\sqrt{q^7}}$$

$$\times[(1-k^2)(2+5k^2-8k^4)K(k)-2(1-2k^2)(1+4k^2-4k^4)E(k)]. \quad (42)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(2axy)x^{-2}y^{-2}dxdy=\frac{2\sqrt{pq}}{9k^2}[(2-3k^2)(1-k^2)K(k)-2(1-2k^2)E(k)]. \tag{43}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)xy\,dxdy=\frac{1}{4pqk^3(1-k^2)}$$
$$\times[(8-7k^2)E(k)-(1-k^2)(8-3k^2)K(k)]. \tag{44}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)xy^3dxdy=\frac{1}{8pq^2k^3(1-k^2)^2}$$
$$\times[(8-9k^2)(1-k^2)K(k)-(8-13k^2+3k^4)E(k)]. \tag{45}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)xy^5dxdy=\frac{1}{16pq^3k^3(1-k^2)^3}$$
$$\times[(1-k^2)(8-15k^2+3k^4)K(k)-(1-2k^2)(8-3k^2+3k^4)E(k)]. \tag{46}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)x^3y^3dxdy=\frac{1}{16p^2q^2k^3(1-k^2)^3}$$
$$\times[(8-23k^2+23k^4)E(k)-(1-k^2)(8-19k^2+15k^4)K(k)]. \tag{47}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)x^3y^5dxdy=\frac{1}{32p^2q^3k^3(1-k^2)^4}$$
$$\times[(8-33k^2+58k^4+15k^6)E(k)-(1-k^2)(8-29k^2+45k^4)K(k)]. \tag{48}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)x^5y^5dxdy=\frac{1}{64p^3q^3k^3(1-k^2)^5}$$
$$\times[(8-47k^2+142k^4+281k^6)E(k)-(1-k^2)(8-43k^2+122k^4+105k^6)K(k)]. \tag{49}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)x^{-1}y\,dxdy=\frac{1}{6qk^3}[(8-5k^2)K(k)-(8-k^2)E(k)]. \tag{50}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)x^{-1}y^3dxdy=\frac{1}{12q^2k^3(1-k^2)}$$
$$\times[(8-3k^2-2k^4)E(k)-(1-k^2)(8+k^2)K(k)]. \tag{51}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)x^{-1}y^5dxdy=\frac{1}{24q^3k^3(1-k^2)^2}$$
$$\times[(1+k^2)(8-13k^2+8k^4)E(k)-(1-k^2)(8-k^2-4k^4)K(k)]. \tag{52}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)x^{-1}y^{-1}dxdy=\frac{1}{9k^3}[(8-7k^2)E(k)-(8-3k^2)(1-k^2)K(k)]. \tag{53}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)x^{-1}y^{-3}dxdy=\frac{2q}{45k^3}[(8-9k^2)(1-k^2)K(k)$$

$$-(8-13k^2+3k^4)E(k)]. \qquad (54)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)x^{-3}y\,dxdy=\frac{p}{15qk^3}$$

$$\times[(8-3k^2-2k^4)E(k)-(8+k^2)(1-k^2)K(k)]. \qquad (55)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)x^{-3}y^3dxdy=\frac{p}{30q^2k^3}[(8+3k^2+4k^4)\ K(k)-(8+7k^2+8k^4)E(k)]. \qquad (56)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)x^{-3}y^5dxdy=\frac{p}{60q^3k^3(1-k^2)}$$

$$\times[(1-k^2)(8+13k^2+24k^4)K(k)-(8+9k^2+16k^4-48k^6)E(k)]. \qquad (57)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)x^{-3}y^{-3}dxdy=\frac{4pq}{225k^3}[(8-23k^2+23k^4)\ E(k)$$

$$-(1-k^2)(8-19k^2+15k^4)K(k)]. \qquad (58)$$

5.2. $k^2=\dfrac{\sqrt{pq}-\sqrt{pq-a^2}}{2\sqrt{pq}}$, $\quad pq>a^2$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0^2(axy)dxdy=\frac{1}{\pi\sqrt{pq}}[K(k)]^2. \qquad (1)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0^2(axy)x^2dxdy=\frac{1}{2\pi(1-2k^2)\sqrt{p^3q}}K(k)\ [2E(k)-K(k)]. \qquad (2)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1^2(axy)dxdy=\frac{1}{\pi k^2(1-k^2)\sqrt{pq}}[E(k)-\ 1-k^2)K(k)]. \qquad (3)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1^2(axy)x^2dxdy=\frac{1}{2\pi k^2(1-k^2)(1-2k^2)\sqrt{p^3q}}$$

$$\times[E(k)-(1-k^2)K(k)][(1-k^2)K(k)-(1-2k^2)E(k)]. \qquad (4)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1^2(axy)x^{-2}y^2dxdy=\frac{\sqrt{p}}{3\pi k^2(1-k^2)\sqrt{q^3}}\times[(1-k^2)K(k^1)-(1-2k^2)E(k)]^2 \qquad (5)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1^2(axy)x^{-2}y^4dxdy=\frac{\sqrt{p}}{6\pi k^2(1-k^2)(1-2k^2)\sqrt{q^5}}[(1-16k^2+16k^4)E(k)$$

$$-(1-k^2)(1-8k^2)K(k)]\ [(1-2k^2)E(k)-(1-k^2)K(k)]. \qquad (6)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_{\frac{2}{3}}^2(axy)\,dxdy=\frac{1}{9\pi k^4(1-k^2)^2\sqrt{pq}}\,[(2-3k^2)$$

$$\times(1-k^2)K(k)-2(1-2k^2)E(k)]^2. \qquad (7)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_{\frac{2}{3}}^2(axy)x^2\,dxdy=\frac{1}{6\pi k^4(1-k^2)^2(1-2k^2)\sqrt{p^3q}}\,[2(1-k^2+k^4)E(k)$$

$$-(2-k^2)(1-k^2)K(k)]\;[(2-3k^2)(1-k^2)K(k)-2(1-2k^2)E(k)]. \qquad (8)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_{\frac{2}{3}}^2(axy)x^{-2}y^2\,dxdy=\frac{\sqrt{p}}{15\pi k^4(1-k^2)^2\sqrt{q^3}}$$

$$\times[2(1-k^2+k^4)E(k)-(1-k^2)(2-k^2)K(k)]^2. \qquad (9)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_{\frac{3}{3}}^2(axy)\,dxdy=\frac{1}{225\pi k^6(1-k^2)^3\sqrt{pq}}$$

$$\times[(8-23k^2+23k^4)E(k)-(1-k^2)(8-19k^2+15k^4)K(k)]^2. \qquad (10)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_{\frac{3}{3}}^2(axy)x^{-2}y^2\,dxdy=\frac{\sqrt{p}}{315\pi k^6(1-k^2)^3\sqrt{q^3}}$$

$$\times\,[(1-2k^2)(8-3k^2+3k^4)E(k)-(1-k^2)(8-15k^2+3k^4)K(k)]^2. \qquad (11)$$

5.3. $k^2=\dfrac{\sqrt{pq}-\sqrt{pq-a^2}}{2\sqrt{pq}}$, $\quad pq>a^2$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1(axy)I_0(axy)xy\,dxdy=\frac{1-2k^2}{2\pi pqk\sqrt{1-k^2}}\,K(k)\,[E(k)-(1-k^2)K(k)]. \qquad (1)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1(axy)I_0(axy)x^{-1}y^3\,dxdy=\frac{1-2k^2}{2\pi q^2k\sqrt{1-k^2}}$$

$$\times\,[2E(k)-K(k)]\,[(1-k^2)K(k)-(1-2k^2)E(k)]. \qquad (2)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(axy)I_1(axy)xy\,dxdy=\frac{1-2k^2}{2\pi pqk^3\sqrt{(1-k^2)^3}}$$

$$\times\,[E(k)-(1-k^2)K(k)]\,[(2-3k^2)(1-k^2)K(k)-2(1-2k^2)E(k)]. \qquad (3)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(axy)I_1(axy)x^{-1}y^3\,dxdy=\frac{1-2k^2}{6\pi q^2k^3\sqrt{(1-k^2)^3}}$$

$$\times\,[(1-k^2)K(k)-(1-2k^2)E(k)]\;[2(1-k^2+k^4)E(k)-(1-k^2)(2-k^2)K(k)]. \qquad (4)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(axy)I_2(axy)xydxdy$$

$$=\frac{1-2k^2}{18\pi pqk^5\sqrt{(1-k^2)^5}}\left[(1-k^2)(2-3k^2)K(k)-2(1-2k^2)E(k)\right]+$$

$$\times\left[(8-23k^2+23k^4)E(k)-(1-k^2)(8-19k^2+15k^4)K(k)\right]. \quad (5)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(axy)I_2(axy)x^{-1}y^3dxdy=\frac{1-2k^2}{30\pi q^2k^5\sqrt{(1-k^2)^5}}\left[2(1-k^2+k^4)E(k)\right.$$

$$\left.-(1-k^2)(2-k^2)K(k)\right]\left[(1-k^2)(8-15k^2+3k^4)K(k)-(1-2k^2)(8-3k^2+3k^4)E(k)\right]. \quad (6)$$

5.4. $k^2=\dfrac{4abc^2}{[(p+a)(q-b)-c^2][(p-a)(q+b)-c^2]}$

$$(p-a)(q-b)>c^2, \qquad p>a, \qquad q>b$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(ax^2)I_0(by^2)I_0(2cxy)\ xydxdy=\frac{k}{4\pi c\sqrt{ab}}K(k). \quad (1)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(ax^2)I_0(by^2)I_0(2cxy)\ xy^3dxdy=\frac{k^3(qp^2-qa^2-pc^2)}{16\pi(1-k^2)c^3\sqrt{(ab)^3}}E(k). \quad (2)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(ax^2)I_1(by^2)I_0(2cxy)\ xy^3dxdy$$

$$=\frac{k^3[(qp^2-qa^2-pc^2)^2+b^2(p^2-a^2)^2-a^2c^4]}{32\pi(1-k^2)(p^2-a^2)c^3\sqrt{a^3b^5}}E(k)-\frac{k}{8\pi c\sqrt{ab^3}}K(k). \quad (3)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1(ax^2)I_1(by^2)I_2(2cxy)\ xydxdy=\frac{1}{4\pi kc\sqrt{ab}}\left[(2-k^2)K(k)-2E(k)\right]. \quad (4)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1(ax^2)I_1(by^2)I_2(2cxy)xy^3dxdy=\frac{k(qp^2-qa^2-pc^2)}{16\pi(1-k^2)c^3\sqrt{(ab)^3}}$$

$$\times\left[(2-k^2)E(k)-2(1-k^2)K(k)\right]. \quad (5)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(ax^2)I_2(by^2)I_4(2cxy)xydxdy=\frac{1}{12\pi k^3c\sqrt{ab}}$$

$$\times\left[(4-k^2)(4-3k^2)K(k)-8(2-k^2)E(k)\right]. \quad (6)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(ax^2)I_3(by^2)I_6(2cxy)xydxdy=\frac{1}{60\pi k^5c\sqrt{ab}}$$

$$\times\left[(128-128k^2+15k^4)(2-k^2)K(k)-2(128-128k^2+23k^4)E(k)\right]. \quad (7)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_4(ax^2)I_4(by^2)I_8(2cxy)xydxdy=\frac{1}{420\pi k^7c\sqrt{ab}}$$

$$\times\left[(6144-12288k^2+8000k^4-1856k^6+105k^8)K(k)-32(2-k^2)(96-96k^2+11k^4)E(k)\right]. \quad (8)$$

5.5. $k^2=\dfrac{\sqrt{pq}-\sqrt{pq-b^2}}{2\sqrt{pq}}$, $\quad pq>b^2$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}K_0(2\sqrt{pq}xy)I_0(4bxy)dxdy=\frac{k\sqrt{1-k^2}}{b}\,[K(k)]^2. \tag{1}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}K_0(2\sqrt{pq}xy)I_0(4bxy)x^2dxdy=\frac{k\sqrt{1-k^2}}{4pb(1-2k^2)}\,K(k)[2E(k)-K(k)]. \tag{2}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}K_0(2\sqrt{pq}xy)I_1(4bxy)xydxdy=\frac{1}{4b\sqrt{pq}(1-2k^2)}\,K(k)[E(k)-(1-k^2)K(k)]. \tag{3}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}K_0(2\sqrt{pq}xy)I_2(4bxy)dxdy=\frac{1}{bk\sqrt{1-k^2}}\,[E(k)-(1-k^2)K(k)]^2. \tag{4}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-by^2}K_0(2\sqrt{pq}xy)I_2(4bxy)x^2dxdy=\frac{1}{4pbk(1-2k^2)\sqrt{1-k^2}}$$
$$\times[E(k)-(1-k^2)K(k)]\;[(1-k^2)K(k)-(1-2k^2)E(k)]. \tag{5}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}K_0(2\sqrt{pq}\,xy)I_3(4bxy)\;xydxdy=\frac{1}{4b\sqrt{pq}\,k^2(1-k^2)(1-2k^2)}$$
$$\times[E(k)-(1-k^2)K(k)]\,[(1-k^2)(2-3k^2)K(k)-2(1-2k^2)E(k)]. \tag{6}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}K_0(2\sqrt{pq}\,xy)I_4(4bxy)\;dxdy=\frac{1}{9bk^3\sqrt{(1-k^2)^3}}$$
$$\times[(1-k^2)(2-3k^2)K(k)-2(1-2k^2)E(k)]^2. \tag{7}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}K_0(2\sqrt{pq}\,xy)I_4(4bxy)\;x^2dxdy=\frac{1}{12pbk^3(1-2k^2)\sqrt{(1-k^2)^3}}$$
$$\times[(2-3k^2)(1-k^2)K(k)-2(1-2k^2)E(k)]\;[2(1-k^2+k^4)E(k)-(1-k^2)(2-k^2)K(k)]. \tag{8}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}K_1(2\sqrt{pq}\,xy)I_0(4bxy)\;xydxdy=\frac{k\sqrt{1-k^2}}{4b\sqrt{pq}\,(1-2k^2)}K(k)[2E(k)-K(k)]. \tag{9}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}K_1(2\sqrt{pq}\,xy)I_1(4bxy)\;x^2dxdy=\frac{1}{4pb(1-2k^2)}\,[2E(k)-K(k)]$$
$$\times[(1-k^2)K(k)-(1-2k^2)E(k)]. \tag{10}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}K_1(2\sqrt{pq}\,xy)I_2(4bxy)\;xydxdy=\frac{1}{4bk(1-2k^2)\sqrt{pq(1-k^2)}}$$
$$\times[E(k)-(1-k^2)K(k)]\times[(1-k^2)K(k)-(1-2k^2)E(k)]. \tag{11}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}K_1(2\sqrt{pq}\,xy)I_2(4bxy)\;x^{-1}ydxdy=\frac{\sqrt{p}}{3bk\sqrt{q(1-k^2)}}$$
$$+[(1-2k^2)E(k)-(1-k^2)K(k)]^2. \tag{12}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}K_1(2\sqrt{pq}\,xy)I_2(4bxy)\;x^{-1}y^3dxdy=\frac{\sqrt{p}}{12bk(1-2k^2)\sqrt{q^3(1-k^2)}}$$

$$\times[(1-k^2)K(k)-(1-2k^2)E(k)]\;[(1-k^2)(1-8k^2)K(k)-(1-16k^2+16k^4)E(k)]. \qquad (13)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}K_1(2\sqrt{pq}\,xy)I_4(4bxy)\;xydxdy=\frac{1}{12bk^3(1-2k^2)\sqrt{pq(1-k^2)^3}}\;[(2-3k^2)$$

$$\times(1-k^2)K(k)-2(1-2k^2)E(k)]\;[2(1-k^2+k^4)E(k)-(1-k^2)(2-k^2)K(k)]. \qquad (14)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}K_2(2\sqrt{pq}\,xy)I_2(4bxy)\;x^2dxdy=\frac{1}{12pbk(1-2k^2)\sqrt{1-k^2}}\;[(1-k^2)K(k)$$

$$-(1-2k^2)E(k)]\;[(1-8k^2)(1-k^2)K(k)-(1-16k^2+16k^4)E(k)]. \qquad (15)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}K_2(2\sqrt{pq}\,xy)I_4(4bxy)\;x^{-2}y^4dxdy=\frac{p}{420bq^2\,k^3(1-2k^2)\sqrt{(1-k^2)^3}}\;[(2+15k^2$$

$$-144k^4+128k^6)(1-k^2)K(k)-2(1+7k^2-135k^4+256k^6-128k^8)E(k)]$$

$$\times[(1-k^2)(2+5k^2-8k^4)K(k)-2(1-2k^2)(1+4k^2-4k^4)E(k)]. \qquad (16)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}K_3(2\sqrt{pq}\,xy)I_4(4bxy)\;x^{-1}y^3dxdy=\frac{\sqrt{p}}{420bk^3(1-2k^2)\sqrt{q^3(1-k^2)^3}}$$

$$\times[(2+15k^2-144k^4+128k^6)\;(1-k^2)K(k)-2(1+7k^2-135k^4+256k^6-128k^8)E(k)]$$

$$\times\;[(1-k^2)(2+5k^2-8k^4)K(k)-2(1-2k^2)(1+4k^2-4k^4)E(k)]. \qquad (17)$$

5.6. $k^2=\dfrac{4pqa}{(pq+a)^2-b^2}$, $pq>a+b$

$$\int_0^\infty\int_0^\infty e^{-p^2x^2-q^2y^2}I_0(2axy)\cosh(2bxy)\;dxdy=\frac{k}{4\sqrt{pqa}}K(k). \qquad (1)$$

$$\int_0^\infty\int_0^\infty e^{-p^2x^2-q^2y^2}I_1(2axy)\cosh(2bxy)\;x^{-1}ydxdy=\frac{\sqrt{p}}{4k\sqrt{q^3a}}\;[(2-k^2)K(k)-2E(k)]. \qquad (2)$$

$$\int_0^\infty\int_0^\infty e^{-p^2x^2-q^2y^2}I_2(2axy)\cosh(2bxy)\;x^{-2}y^2dxdy=\frac{\sqrt{p^3}}{12k^3\sqrt{q^5a}}$$

$$\times[(4-k^2)(4-3k^2)K(k)-8(2-k^2)E(k)]. \qquad (3)$$

$$\int_0^\infty\int_0^\infty e^{-p^2x^2-q^2y^2}I_3(2axy)\cosh(2bxy)\;x^{-3}y^3dxdy=\frac{\sqrt{p^5}}{60k^5\sqrt{q^7a}}$$

$$[(2-k^2)(128-128k^2+15k^4)K(k)-2(128-128k^2+23k^4)E(k)]. \qquad (4)$$

$$\int_0^\infty\int_0^\infty e^{-p^2x^2-q^2y^2}I_4(2axy)\cosh(2bxy)\ x^{-4}y^4dxdy=\frac{\sqrt{p^7}}{420k^7\sqrt{q^9a}}$$

$$\times[(6144-12288k^2+8000k^4-1856k^6+105k^8)K(k)-32(2-k^2)(96-96k^2+11k^4)E(k)]. \quad (5)$$

$$\int_0^\infty\int_0^\infty e^{-p^2x^2-q^2y^2}I_0(2axy)\sinh(2bxy)\ xydxdy=\frac{bk^3}{32(1-k^2)\sqrt{(pqa)^3}}E(k). \quad (6)$$

$$\int_0^\infty\int_0^\infty e^{-p^2x^2-q^2y^2}I_1(2axy)\sinh(2bxy)\ x^2dxdy=\frac{bk}{32(1-k^2)\sqrt{p^5qa^3}}$$

$$\times[(2-k^2)E(k)-2(1-k^2)K(k)]. \quad (7)$$

$$\int_0^\infty\int_0^\infty e^{-p^2x^2-q^2y^2}I_2(2axy)\sinh(2bxy)\ x^{-1}y^3dxdy=\frac{b\sqrt{p}}{32k(1-k^2)\sqrt{q^7a^3}}$$

$$\times\ [(16-16k^2+k^4)E(k)-8(1-k^2)(2-k^2)K(k)]. \quad (8)$$

$$\int_0^\infty\int_0^\infty e^{-p^2x^2-q^2y^2}I_3(2axy)\sinh(2bxy)\ x^{-2}y^4dxdy=\frac{b\sqrt{p^3}}{96k^3(1-k^2)\sqrt{q^9a^3}}$$

$$\times\ [(2-k^2)(128-128k^2+3k^4)E(k)-2(1-k^2)(128-128k^2+27k^4)K(k)]. \quad (9)$$

5.7. $\quad k_1^2=\dfrac{(pq-a)^2-b^2}{(pq+a)^2-b^2}, \qquad k_2^2=\dfrac{b^2-(pq-a)^2}{4pqa}$

$$\int_0^\infty\int_0^\infty e^{-p^2x^2-q^2y^2}K_0(2axy)\cosh(2bxy)\ dxdy=\frac{\pi\sqrt{1-k_1^2}}{4\sqrt{pqa}}K(k_1), \qquad b<|pq-a|;$$

$$=\frac{\pi}{4\sqrt{pqa}}K(k_2), \qquad |pq-a|<b<pq+a. \quad (1)$$

$$\int_0^\infty\int_0^\infty e^{-p^2x^2-q^2y^2}K_0(2axy)\sinh(2bxy)\ xydxdy$$

$$=\frac{\pi b\sqrt{(1-k_1^2)^3}}{32k_1^2\sqrt{(pqa)^3}}[K(k_1)-E(k_1)], \qquad b<|pq-a|;$$

$$=\frac{\pi b}{32k_2^2(1-k_2^2)\sqrt{(pqa)^3}}[E(k_2)-(1-k_2^2)K(k_2)], \qquad |pq-a|<b<pq+a. \quad (2)$$

$$\int_0^\infty \int_0^\infty e^{-p^2x^2-q^2y^2} K_1(2axy)\sinh(2bxy)\; x^2dxdy$$

$$= \frac{\pi b\sqrt{1-k_1^2}}{32k_1^2\sqrt{p^5qa^3}} [(1+k_1^2)E(k_1)-(1-k_1^2)K(k_1)], \quad b<|pq-a|;$$

$$= \frac{\pi b}{32k_2^2(1-k_2^2)\sqrt{p^5qa^3}} [(1-k_2^2)K(k_2)-(1-2k_2^2)E(k_2)],$$

$$|pq-a|<b<pq+a. \qquad (3)$$

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6. References

For preparation of this paper the author referred to general expressions of these types of integrals in terms of hypergeometric functions or Legendre functions. The section numbers and the general expressions referred there are:

2.1 OF 63,167, OB 50 (6.10),(6.11), ET I 45 (13), 100 (11); 2.2 OF 65,170; 2.4 OF 65; 2.5 OF 169; 2.7 OF 69; 2.8 OF 69; 2.10 OF 174, ET II 348 (65); 2.12 OF 87,184, ET I 49 (42),106 (50); 2.13 OF 89,185, ET I 107 (59),(60); 2.14 OF 88,185; 2.15 OF 89,186; 2.16 OF 88,185; ET I 50 (51),107 (61); 2.18 OF 88,185, ET I 49 (47),106 (54); 2.19 LU 315 (13); 2.20 OF 88,185; 4.2 OB 125 (14.11); 4.4 EG 18 (19); 4.5 ET II 410 (43); 4.6 ET II 387 (13); 4.7 ET II 370 (44); 4.8 ET II 325 (12); 4.9 ET II 351 (10); 4.10 ET II 352 (15); 5.1 NM II 33 (137); 5.6 OS 624 (7).

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$$\lim_{a \to 0^+} \int_a^1 f(x)dx$$

exists and is finite. There are computational schemes that produce a convergent sequence of approximations to the integral of any function in R; the trapezoid rule is one. In this paper, it is shown that there is no computational scheme that uses only evaluations of the integrand, that is similarly effective for IR.

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Key words: asymptotic approximations; error bounds; Legendre functions; parabolic cylinder functions; turning points; Weber functions.

By application of the theory for second-order linear differential equations with two turning points developed in the preceding paper, some new asymptotic approximations are obtained for the associated Legendre functions when both the degree n and order m are large. The approximations are expressed in terms of parabolic cylinder functions, and are uniformly valid with respect to $x \in (-1,1)$ and $m/(n+1/2) \in [\delta, 1+\Delta]$, where δ and Δ are arbitrary fixed numbers such that $0 < \delta < 1$ and $\Delta > 0$. The values of m and $n+1/2$ are either both real, or both purely imaginary. In all cases explicit bounds are supplied for the error terms associated with the approximations.

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Key words: asymptotic approximations; connection formulas; error bounds; ordinary differential equations; parabolic cylinder functions; turning points; wave scattering; Weber functions.

Differential equations of the form

$$d^2w/dx^2 = \{u^2f(u,a,x)+g(u,a,x)\}w$$

are considered for large values of the real parameter u. Here x is a real variable ranging over an open, possibly infinite, interval (x_1, x_2), and a is a bounded real parameter. It is assumed that $f(u,a,x)$ and $g(u,a,x)$ are free from singularity within (x_1, x_2), and $f(u,a,x)$ has exactly two zeros, which depend continuously on a and coincide for a certain value of a. Except in the neighbourhoods of the zeros, $g(u,a,x)$ is small in absolute value compared with $u^2f(u,a,x)$.

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